First and Second Order Transient Circuits

When conditions in a circuit change, for example when you turn it on or switch operating states, its behavior is transient. Circuit parameters change over some time allowing the change of state, then approach steady state operation.

Note periodic signals are changing over time, but since conditions repeat at regular intervals, the circuit reaches a periodic (AC) steady state.

At this stage, to analyze transients we will work in the time domain -> with equations including t or derivatives d/dt.

Consider a camera flash operating from a battery:

The battery charges a capacitor (fairly slowly). When the switch is moved (depressed) the capacitor discharges rapidly through the lamp.

We may be interested in the charging and discharging process:

Consider the discharge circuit:
When the switch is closed, KCL at a node between R and C:

\[ C \frac{dv_c(t)}{dt} + \frac{v_c(t)}{R_L} = 0 \]

\[ \frac{dv_c(t)}{dt} + \frac{1}{CR_L} v_c(t) = 0 \]

Which has a solution

\[ v_c(t) = v_o e^{-t/RC} \]

Exponential charging with a rate that depends on the RC product.

**First Order Transients**

In general, the problem will have the form:

\[ \frac{dx(t)}{dt} + ax(t) = f(t) \]

And the solution can be a linear combination of forced response (including f(t)) and natural response (with f(t) = 0) so:

\[ x(t) = x_p(t) + x_c(t) \]

(particular and complementary)

If we limit our cases to a constant for the forcing function (A), then a solution can also be a constant \( K_1 \):

\[ \frac{dx_p(t)}{dt} + ax_p(t) = A \]

\[ x_p(t) = K_1 \]

For the homogeneous equation (=0)

\[ \frac{dx_c(t)}{dt} + ax_c(t) = 0 \]

\[ \frac{1}{x_c(t)} \frac{dx_c(t)}{dt} = -a \]

You may recognize a solution
\[ \ln(x_c(t)) = -at + c \]

Or

\[ x_c(t) = K_2 e^{-at} \]

And we can put together a general solution:

\[ x(t) = K_1 + K_2 e^{-\frac{t}{\tau}} \]

Where \( K_1 \) is the steady state solution and \( \tau \) is the time constant.

And we can solve problems by fitting our initial and final conditions or by writing and solving D.E.s.

**Transient Circuit Analysis**

- differential equation approach

\( \rightarrow \) write equation for voltage on a capacitor or current through an inductor.

Note: These quantities don’t change instantaneously.

Write KCL:

\[
C \frac{dv(t)}{dt} + \frac{v(t) - V_s}{R} = 0
\]

Or

\[
\frac{dv(t)}{dt} + \frac{v(t)}{RC} = \frac{V_s}{RC}
\]

Which should have a solution

\[ v(t) = K_1 + K_2 e^{-\frac{t}{\tau}} \]

Substitute this into the differential equation:

\[
\left( -\frac{K_2}{\tau} e^{-\frac{t}{\tau}} \right) + \frac{1}{RC} \left( K_1 + K_2 e^{-\frac{t}{\tau}} \right) = \frac{V_s}{RC}
\]

Noting that the constant terms must be equal:

\[
\frac{K_1}{RC} = \frac{V_s}{RC}
\]
Or $K_1 = V_S$

The exponential terms must also be equal so:

$$\frac{K_2}{\tau} = \frac{K_2}{RC}$$

Or $\tau = RC$

We still need $K_2$ -> consider the initial condition

$V(0) = 0V = K_1 + K_2 = V_s + K_2$

$K_2 = -V_s$

And

$$v(t) = V_S - V_S e^{-\frac{t}{\tau}} = V_S \left(1 - e^{-\frac{t}{\tau}}\right)$$

If we had an inductor instead

Write KVL:

$$L \frac{di(t)}{dt} + Ri(t) = V_S$$

And

$$i(t) = K_1 + K_2 e^{-\frac{t}{\tau}}$$

$$\left(\frac{-K_2}{\tau} e^{-\frac{t}{\tau}}\right) + \frac{R}{L} \left(K_1 + K_2 e^{-\frac{t}{\tau}}\right) = \frac{V_S}{L}$$

Constant terms:

$$\frac{K_1 R}{L} = \frac{V_S}{L}$$

$$K_1 = \frac{V_S}{R}$$

Exponential terms:

$$\frac{K_2}{\tau} = \frac{K_2 R}{L}$$
\[ \tau = \frac{L}{R} \]

And at \( t = 0 \)

\[ i(0) = 0 = \frac{V_S}{R} + K_2 \]

\[ K_2 = -\frac{V_S}{R} \]

So

\[ i(t) = \frac{V_S}{R} \left( 1 + e^{-\frac{R}{\tau} t} \right) \]

And if we want \( v_R(t) \):

\[ v_R(t) = R \cdot i(t) = V_S \left( 1 + e^{-\frac{R}{\tau} t} \right) \]

(similar to capacitor voltage in RC).

**Transient Circuit Analysis Step by Step Approach**

We still expect a solution

\[ x(t) = K_1 + K_2 e^{-\frac{t}{\tau}} \]

(\( x \) might be current or voltage)

Note as \( t \to \infty \) \( e^{-\frac{t}{\tau}} \to 0 \) \( x(t) \to K_1 \)

So if we find \( x(t) \) in steady-state (\( t \to \infty \)) with capacitor open circuit (for DC) and inductors short circuit (for DC), then \( x(t) = K_1 \).

Note at \( t = 0 \) \( x(t) = K_1 + K_2 \)

Or at some time \( t_0 \):

\[ x(t_0) = K_1 + K_2 e^{-\frac{t_0}{\tau}} \]

So if we know the initial value \( x(0) \) or the value at some time \( x(t_0) \) we can also find \( K_2 \).

Often the ‘initial condition’ is just after a switch moves and is determined by the circuit before the switch changed.

The time constant can be found from the Thevenin equivalent resistance at the terminals of \( C \) or \( L \).

\[ \tau = R_{th} C \text{ or } \tau = \frac{L}{R_{th}} \]
Example Find \(i(t)\)  \(t > 0\):

1) We expect: \(v_c(t) = K_1 + K_2 e^{\frac{-t}{\tau}}\)

2) Initial capacitor voltage \(t = 0\) (steady state open circuit)
   - \(i(0^-) = (36V-12V)/(2k+6k+4k) = 2mA\)
   - \(v_c(0^-) = 36V - 2mA \times 2k = 32V\)
   - Note \(i\) will change as soon as switch moves. \(v_c\) must be continuous.

3) At \(t=0^+\) the switch connects 6k to ground. So new circuit is:

4) as \(t \to \infty\) the capacitor is again open circuit
   - \(i(\infty) = 36V/(6k+2k) = 4.5mA\)
   - \(v_c(\infty) = 4.5mA \times 6k = 27V = K_1\)

5) From the initial condition we know that:
   - \(v_c(0^-) = K_1 + K_2 e^0 = 32V \rightarrow K_2 = 32V - 27V = 5V\)

6) To find \(\tau\) we need the equivalent resistance seen by the capacitor
   - Note the 4k resistor is still shorted out by the switch.
   - Therefore \(R_{th} = 6k/2k = 1.5k\)
   - So \(\tau = R_{th} \cdot C = 1.5k \times 100\mu F = 0.15s\)
   - And our solution \(v_c(t) = \left(27 + 5e^{\frac{-t}{0.15}}\right)V\)
   - The current is then easily found by ohm’s law as: \(i(t) = \frac{v_c(t)}{6k} = 4.5 + 0.83e^{\frac{-t}{0.15}}mA\)
Note the switch doesn’t always change at \( t = 0 \) -> we may have an \( e^{-\frac{t}{\tau}} \) term.

- e.g. \( x(t) = x(\infty) + [x(t_o) - x(\infty)]e^{-\frac{(t-t_o)}{\tau}} \)
- be very careful applying formula solutions!

**Pulse Response**

Many circuits are characterized by their response to a unit impulse or unit step function. We’ll just consider a step for now

Unit step function \( u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \)

And a more general form: \( u(t) = \begin{cases} 0 & t < t_o \\ 1 & t > t_o \end{cases} \)

Note steps can be used to construct other functions, such as pulses:

\[
    v(t) = A[u(t) - u(t - T)]
\]

is a pulse from 0 to \( T \)

\[
    v(t) = A[u(t - t_o) - u(t - (t_o + T))]\]

is a pulse from \( t_o \) to \( t_o + T \)

Note we also write such functions piece wise.
Example what is \( v_o(t) \) \( t > 0 \)?

\[ v_o(t) = 0 \] for \( t < 0 \) (no source)

at \( t = 0^+ \) \( i_o = 0 \) (inductor) so \( v_o = 0V \)

as \( t \to \infty \) if pulse did not end, inductor = short, \( v_o = 1/3 \times 12V = 4V \)

\[ R_{th} = 3 \Omega \]

***Note this is not obvious to a lot of students, but the 2ohm resistor on the right is in series with the other two in parallel. Remember a circuit is a circle!***

so \( \tau = \frac{L}{R} = \frac{2}{3} = 0.67s \)

\[ v_o(t) = K_1 + K_2 e^{-\frac{t}{\tau}} \]

\( t \to \infty \) \( K_1 = 4V \)

\( t \to 0 \) \( v_o(t) = 0 = K_1 + K_2 \to K_2 = -K_1 = -4V \)

So

\[ v_o(t) = 4 \left(1 - e^{-\frac{3t}{2}}\right) \quad 0 < t < 1s \]

at \( t = 1s \) \( v_o(1) = 4 \left(1 - e^{-\frac{3}{2}}\right) = 3.11V \)

Now the pulse is turned off so eventually the voltage returns to zero. Therefore:

\[ v_o(\infty) = 0 = K_1 \]
The initial condition again gives us $K_2$ directly since $K_1$ is zero as $3.11V$.

$$\text{so} \quad v_o(t) = 3.11e^{-\frac{3}{2}(t-1)} \quad \text{for} \ t > 1s$$

You may want to read RLC – we will come back to this later!

**Design Example**

One application of capacitors is to smooth-out sudden voltage changes, since voltage in a capacitor cannot change instantaneously.

Capacitors are widely used for decoupling power supplies -> that is protecting circuits from rapid voltage fluctuations or disturbances.

Say we have a DC voltage source:

![Image of circuit diagram]

We would like a simple circuit to isolate a load from supply fluctuations.

We can do this with a single capacitor across the load:

![Image of circuit diagram with capacitor]

Say our supply changes from $V_s \rightarrow V_s + \Delta V_s$ at time $t = 0$, and returns to $V_s$ at $t = t'$.
How will $C_D$ influence the fluctuation in output voltage $\Delta v_o$?

Voltage on $C_D$: $v_o(t) = K_1 + K_2 e^{-t/\tau}$  

at $t = 0$  $v_o(0) = K_1 + K_2 = V_S - I_L R_S$  

at $t = \infty$  $v_o(\infty) = K_1 = V_S + \Delta V_S - I_L R_S$ (for a step) used to get constants during a pulse  

And the time constant is just $R_S C_D$, so:  

$$v_o(t) = (V_S + \Delta V_S - I_L R_S) - \Delta V_S e^{-t R_S C_D}$$

This applies until $t'$, when the voltage supply returns to $V_S$.  

$$v_o(t') = (V_S + \Delta V_S - I_L R_S) - \Delta V_S e^{-t' R_S C_D}$$

$$= V_S - I_L R_S + \Delta V_o$$

Or  

$$\Delta V_S - \Delta V_o = \Delta V_S e^{-t' R_S C_D}$$  

$$\frac{\Delta V_S}{\Delta V_S - \Delta V_o} = e^{t' R_S C_D}$$

$$R_S C_D = \frac{t'}{\ln \left( \frac{\Delta V_S}{\Delta V_S - \Delta V_o} \right)}$$

Or  

$$C_D = \frac{t' R_S}{\ln \left( \frac{\Delta V_S}{\Delta V_S - \Delta V_o} \right)}$$

Note the choice of $C_D$ depends on voltage change, not magnitude.
Given the expected size and length of voltage disturbances, source R and desired fluctuation in load voltage, we can choose $C_D$.

However good decoupling (\( \frac{\Delta V_o}{\Delta V_s} \textit{small} \)) requires large $C_D$ value.