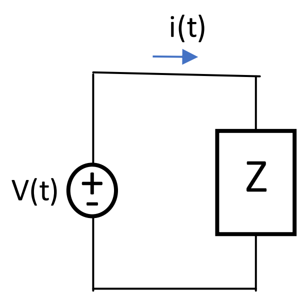
Power Analysis

Instantaneous Power:



We learned previously:

$$p(t) = v(t) \cdot i(t)$$

And for sinusoidal steady state

$$v(t) = V_M \cos(\omega t + \theta_v)$$

$$i(t) = I_M \cos(\omega t + \theta_i)$$

So

$$p(t) = V_M I_M \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

And we can use

$$\cos(\phi_1) \cos(\phi_2) = \frac{1}{2}(\cos(\phi_1 - \phi_2) + \cos(\phi_1 + \phi_2))$$

To write

$$p(t) = \frac{V_M I_M}{2}[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)]$$

The first term is a constant and the second term is time varying.

Average Power:

We also found:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)dt$$

Or for our sinusoid:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} V_M I_M \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)\, dt$$

t_0 is an arbitrary time.

$T = \frac{2\pi}{\omega}$ is the period of voltage or current (and we may average over any number of periods) using the cos product formula above:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_M I_M}{2}[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)]dt$$

Second term averages to zero so

$$P = \frac{V_M I_M}{2}\cos(\theta_v - \theta_i)$$

note $\cos(\theta) = \cos(-\theta)$

for a resistive circuit $\theta_v = \theta_i$

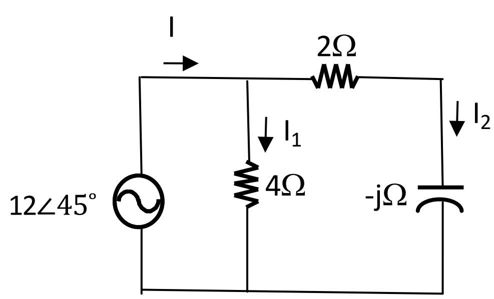
$$P = \frac{V_M I_M}{2}$$

For a reactive circuit

$$P = \frac{V_M I_M}{2}\cos(\pm 90°) = 0$$

***Note a purely reactive circuit does not consume any power!!!!**

Example: Find the power supplied and absorbed



Ohm's Law:

$$I_1 = \frac{12\angle 45^\circ}{4} = 3\angle 45^\circ A$$

$$I_2 = \frac{12\angle 45^\circ}{2 - j} = \frac{12\angle 45^\circ}{2.24\angle -26.6^\circ} = 5.36\angle 71.6^\circ A$$

So the total source current is:

$$I_1 + I_2 = 3\angle 45^\circ + 5.36\angle 71.6^\circ$$

$$= 2.12 + 2.12j + 1.69 + 5.08j = 3.81 + 7.2j = 8.15\angle 62.1^\circ$$

With all currents we can find the power to each element.

Total:

$$P = \frac{V_M I_M}{2}\cos(\theta_v - \theta_i)$$

$$= \frac{12 \cdot 8.15}{2}\cos(45 - 62.1)$$

$$= 46.7W$$

For the 4ohm resistor:

$$P_4 = \frac{V_M I_M}{2} = \frac{12 \cdot 3}{2} = 18W$$

For the 2 ohm resistor we didn't find a value for the voltage so:

$$P_4 = \frac{{I_M}^2 R}{2} = \frac{5.34^2 \cdot 2}{2} = 28.7W$$

So the total $P=P_2+P_4$ = 18+28.7=46.7W which agrees with the first result.

Note if you have more than one source current or voltage can be found by superposition, power cannot.

$$I_T = I_1 + I_2$$

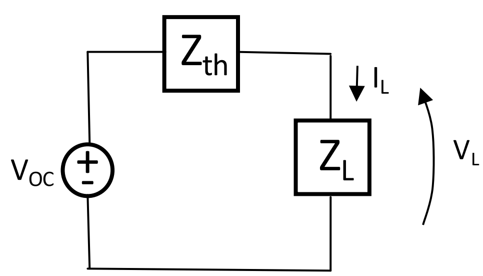
$$P_T = (I_1 + I_2)^2 R \neq {I_1}^2 R + {I_2}^2 R$$

Maximum Average Power Transfer

Recall the thevenin circuit helped us find a condition for maximum power transfer to a resistive load.

For a sinusoidal source driving a complex load:

$$P_L = \frac{V_L I_L}{2}\cos(\theta_v - \theta_i)$$



$$I_L = \frac{V_{OC}}{Z_{TH} + Z_L}$$

$$V_L = \frac{V_{OC} Z_L}{Z_{TH} + Z_L}$$

Where $Z_{TH} = R_{TH} + jX_{TH}$, and $Z_L = R_L + jX_L$

So magnitude

$$I_M = \frac{V_{OC}}{\sqrt{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2}}$$

$$V_M = \frac{V_{OC}\sqrt{{R_L}^2 + {X_L}^2}}{\sqrt{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2}}$$

And $\cos(\theta_v - \theta_i) = \cos(\theta_{Z_L}) = \frac{R_L}{\sqrt{{R_L}^2+{X_L}^2}}$

So

$$P_L = \frac{1}{2}\cdot\frac{{V_{OC}}^2 R_L}{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2}$$

But the reactance doesn't absorb power, and any non zero value $X_{TH} + X_L$ reduces P_L. Therefore $X_{TH} = -X_L$ is the optimum. Then:

$$P_L = \frac{1}{2}\cdot\frac{{V_{OC}}^2 R_L}{(R_{TH}+R_L)^2 + (0)^2}$$

And we have solved this problem for the resistive case.

So we want $R_{TH} = R_L$ and $X_{TH} = -X_L$ or

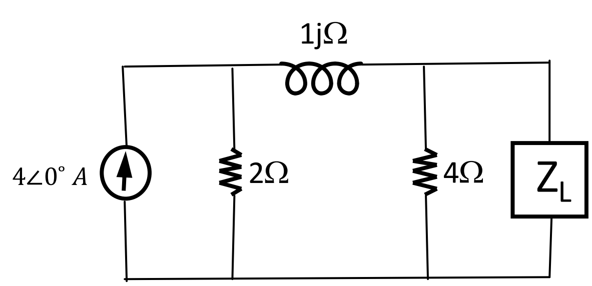
$$Z_L = R_L + jX_L = R_{TH} - jX_{TH} = {Z_{TH}}^*$$

The complex conjugate.

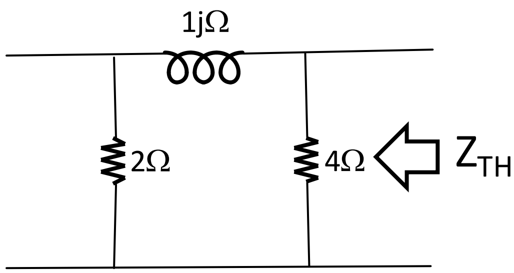
Note if $X_L = 0$ then the value R_L can be found from $\frac{dP_L}{dR_L} = 0$

$$R_L = \sqrt{{R_{TH}}^2 + {X_{TH}}^2}$$

Example: Find Z_L for maximum average power transfer in the following circuit:



We can find a Thevenin equivalent impedance:



$$Z_{TH} = (2 + j)//4$$

$$Z_{TH} = \frac{4(2 + j)}{4 + 2 + j} = \frac{4(2 + j)}{6 + j} = 4 \cdot \frac{2.24\angle 26.6°}{6.08\angle 9.46°} = 1.47\angle 17.14° = 1.41 + 0.43j$$

So we would like:

$$Z_L = 1.41 - 0.43j = {Z_{TH}}^*$$

And we could find the power delivered using a current divider from

$$V_{OC} = 4 \cdot \frac{2 \cdot 4\angle 0°}{6 + j} = 5.26\angle - 9.46°$$

And

$$I = \frac{V_{OC}}{2.82} = 1.87\angle - 9.46°$$

Note: $Z_L + Z_{TH} = 2.82$

So

$$P_L = \frac{1}{2} \cdot {I_M}^2 R_L = \frac{1}{2}(1.87)^2 \cdot 1.41 = 2.47W$$

<u>Effective or RMS Values</u>

Recall early in the course we evaluated RMS average values of periodic signals.

Now we will examine how RMS values relate to the 'effectiveness' of signals at delivering power.

We can define an effective value of a periodic current as the constant DC value that would deliver the same average power to a resistor.

So an effective value I_{eff} would deliver power:

$$P = {I_{eff}}^2 R$$

And a periodic current would deliver an average

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t) R dt$$

Equating the two:

$$I_{eff}^{\ 2} = \frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt$$

Or

$$I_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2(t)dt}$$

Which is the square root of the mean of the current squared or RMS value $I_{eff} = I_{RMS}$

The most common signals we are interested in is the sinusoid. (Note for DC I_{RMS} = I)

For

$$i(t) = I_M \cos(\omega t - \theta)$$

$$I_{RMS} = \sqrt{\frac{1}{T}\int_0^T I_M^{\ 2}\cos^2(\omega t - \theta)\,dt}$$

Note that:

$$\cos^2(\phi) = \frac{1}{2}(1 + \cos 2\phi)$$

$$I_{RMS} = I_M\sqrt{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{1}{2}(1 + \cos(2\omega t - 2\theta))dt}$$

Since the cosine averages to zero we are left with:

$$I_{RMS} = I_M\sqrt{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{1}{2}dt}$$

So

$$I_{RMS} = I_M\sqrt{\frac{\omega}{2\pi}\frac{1}{2}\left(\frac{2\pi}{\omega} - 0\right)} = \frac{I_M}{\sqrt{2}}$$

So a sinusoid with peak current of IM delivers the same average power as a DC current $\frac{I_M}{\sqrt{2}}$

Similarly, for a sinusoid $V_{RMS} = \frac{V_M}{\sqrt{2}}$

Recall average power

$$P = \frac{V_M I_M}{2}\cos(\theta_v - \theta_i)$$

$$= \frac{\sqrt{2}V_{RMS}\sqrt{2}I_{RMS}}{2}\cos(\theta_v - \theta_i)$$

$$P = V_{RMS}I_{RMS}\cos(\theta_v - \theta_i)$$

What is the peak voltage at a wall outlet?

$$V_{line} = 120V_{RMS} = 120\sqrt{2} = 170V$$

$$\omega_{line} = 60Hz \cdot 2\pi = 377rad/s$$

$$v(t) = 120\sqrt{2}\cos(377t)$$

When RMS values are used, it should be indicated in units!

<u>Power Factor</u>

Notice that average power from a sinusoid $P = V_{RMS}I_{RMS}\cos(\theta_v - \theta_i)$depends on the phase factor $\cos(\theta_v - \theta_i)$ which can vary from 0 (reactance) to 1 (resistance). The dimensionless factor $\cos(\theta_v - \theta_i)$ is referred to as <u>power factor</u> and can be defined as the ratio of average power to apparent power. If we neglected phase information (for example by measuring I and V with a multimeter) we would say $P = V_{RMS}I_{RMS}$ which is the <u>apparent</u> power in the system. Therefore power factor is:

$$PF = \frac{V_{RMS}I_{RMS}\cos(\theta_v - \theta_i)}{V_{RMS}I_{RMS}} = \cos(\theta_v - \theta_i)$$

Also notice that from Ohm's law V = IZ so $\theta_v - \theta_i = \theta_Z$ the impedance phase angle so

$$PF = \cos(\theta_{Z_L}) = \cos(-\theta_{Z_L})$$

Since the cosine includes the sign of the angle, P.F. is said to be leading or lagging to phase of current w.r.t. voltage.

For an RC load $Z_L = R - \frac{j}{\omega C}$ and $\theta_{Z_L} < 0$

Since V = ZI current will <u>lead</u> voltage.

RC loads have a leading P.F.

For an RL load load $Z_L = R + j\omega L$ and $\theta_{Z_L} > 0$

Since V = ZI current will <u>lag</u> voltage.

RL loads have a lagging P.F.

P.F. has important economic impact on power systems. Although apparent power may not be dissipated, it does need to be generated and transmitted. At the load it may be absorbed by a resistive load, or simply stored in a reactive load, however there are transmission losses in both cases. P.F. close to 1 is optimal.

Example

A load consumes 88kW at P.F. 0.707 lagging from a 480V_{RMS} line.

Transmission line resistance is 0.08Ω
How much power must be supplied?

$$P = V_{RMS} I_{RMS} PF$$

So $I_{RMS} = \frac{88kW}{480(0.707)} = 259.3A$

Power supplied $P_S = P_L + {I_{RMS}}^2 R = 88kW + 0.08(259.3)^2 = 93.38kW$

But if P.F. is raised to 0.9

$$I_{RMS} = \frac{88kW}{480(0.90)} = 203.7A$$

$$P_S = 88kW + 0.08(203.7)^2 = 91.32kW$$

And line losses are reduced from 5.38kW to 3.32kW.

Note it is often worthwhile to correct power factor (adjust closer to 1) by adding reactance to the load. In microwave engineering we call this impedance matching.

Also, large loads often operate at higher voltage to reduce I^2R losses.

Complex Power

We found real average power from a sinusoidal signal

$$P = \frac{V_M I_M}{2} \cos(\theta_v - \theta_i) = V_{RMS} I_{RMS} \cos(\theta_v - \theta_i)$$

If we represent v(t) and i(t) as phasers, then VI will give us a complex quantity.

To get agreement with average power, we find we need to use

$$S = V_{RMS} {I_{RMS}}^*$$

Where ${I_{RMS}}^*$ is the complex conjugate of I_{RMS}. Then,

$$S = V_{RMS}\angle\theta_v I_{RMS}\angle{-\theta_i}$$

$$= V_{RMS} I_{RMS}\angle\theta_v - \theta_i$$

And we can write this in rectangular coordinates as

$$S = V_{RMS} I_{RMS}(\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i))$$

S is called complex power and of course

$$S = P + jQ$$

We have seen the real term before, it is our real or average power $V_{RMS}I_{RMS}\cos(\theta_v - \theta_i)$ the imaginary term $V_{RMS}I_{RMS}\sin(\theta_v - \theta_i)$ is the reactive power, and the magnitude $|S| = V_{RMS}I_{RMS}$ is what we previously called apparent power.

To keep these different powers distinct

$$P \rightarrow [W]$$

$$S \rightarrow [VA]$$

$$Q \rightarrow [VAR]$$

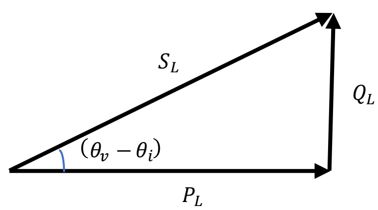
VAR = volt amp reactive.

Inductor $(\theta_v - \theta_i) = 90° \quad Q > 0$

Capacitor $(\theta_v - \theta_i) = -90° \quad Q < 0$

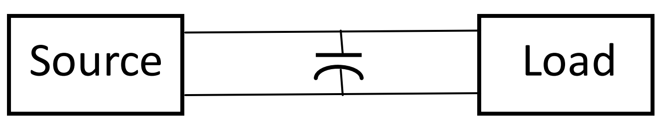
With one calculation we can find real power P dissipated by resistive elements and reactive power Q storing energy in L and C.

Say we have an industrial load with a lagging power factor (current lags voltage $(\theta_v - \theta_i) = \theta_{Z_L} > 0$)



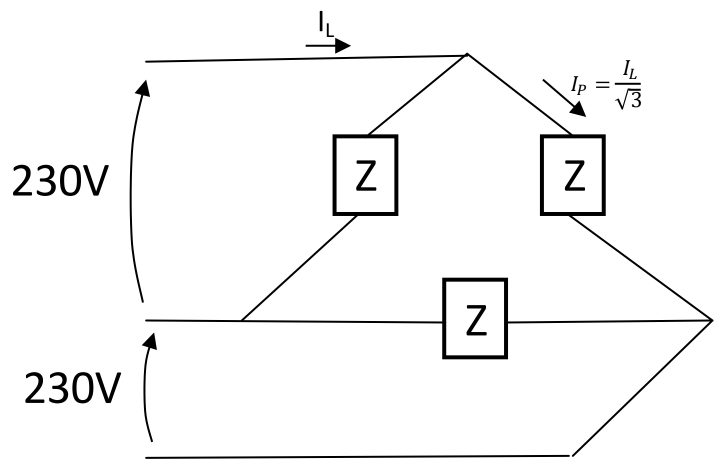
Reduce $(\theta_v - \theta_i)$ and $S_L \rightarrow P_L$ P.F. ->1

To do this, add negative Q -> capacitive load.



Example

A 230V_{RMS} 3ϕ system supplies 2000W to a delta connected balanced load, P.F. 0.9 lagging. Find line current, phase current



Power $P = 3P_\phi$

$= 3 \cdot V \cdot I \cdot PF$

$= 3 \cdot V_L \cdot \frac{I_L}{\sqrt{3}} \cdot PF$

So $2000W = I_L \cdot \sqrt{3} \cdot 230 \cdot 0.9$

$I_L = 5.58A_{rms}$ so $I_\phi = \frac{I_L}{\sqrt{3}} = 3.22A_{rms}$

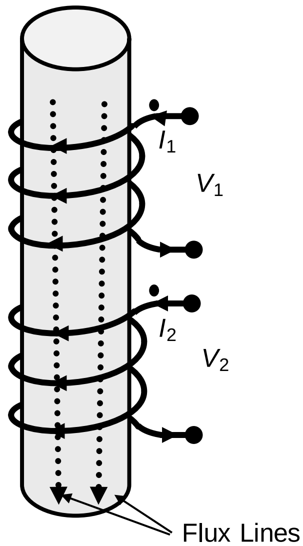
Then $Z = \frac{V}{I} = \frac{|V|}{|I|}\angle(\theta_v - \theta_i) = \frac{230}{3.22}\angle \cos^{-1} 0.9 = 71.4\angle 25.8\Omega$

<u>Mutual Inductance</u>

You will find electrical circuits, magnetic fields, and electromagnetic phenomena are closely linked, and we will develop theory, techniques in ELEC 3105 and 3909.

Amperes Law --- flow of current creates a magnetic field.

Fareday's Law --- time varying magnetic field 'induces' a voltage in a conductor.



In a coil (inductor) the effect is significant as each loop is magnetically coupled.

In a coil with N turns carrying a current i, there is a 'flux linkage' $\lambda = N\phi$ where ϕ is magnetic flux.

Also $\lambda = Li$, so $\phi = \frac{L}{N} \cdot i$

Faraday's law says voltage $v = \frac{d\lambda}{dt}$ or

$$v = L\frac{di}{dt} + i\frac{dL}{dt}$$

Note that usually L is constant and the second term is zero so $v = L\frac{di}{dt}$ which is our familiar expression.

Now if we have a second coil and bring it close enough to the first we can create flux linkage from coil 1 to 2. So if coil 1 has flux linkage

$$\lambda_1 = N_1\phi = L_1 i_1$$

And we can couple all the flux to coil 2 with N_2 turns,

$$\lambda_2 = N_2\phi = N_2\frac{L_1 i_1}{N_1}$$

And induced voltage

$$v_2 = \frac{d}{dt}\left(\frac{N_2}{N_1}L_1 i_1\right) = \frac{N_2}{N_1}L_1\frac{di_1}{dt} = L_{21}\frac{di_1}{dt}$$

And current in coil 1 creates voltage in coil 2.

L_{21} = mutual inductance.

➔ Magnetically coupled coils.

We could have a current in both coils, so

$$\lambda_1 = L_1 i_1 + L_{12} i_2$$

$$\lambda_1 = L_{21} i_1 + L_2 i_2$$

So

$$v_1 = L_1 \frac{di_1}{dt} + L_{12} \frac{di_2}{dt}$$

$$v_2 = L_{21} \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

And for linear circuit $L_{12} = L_{21} = M$.

Note signs depend on current directions.

Dots are used --- both currents into dots -> signs positive

---- one current in, one out - M-terms negative

Or current into dot produces voltage in coupled coil positive at dot.

You can do lots of analysis on different coupled circuits, but the most common application is the transformer.

Faraday $v = \frac{d\lambda}{dt}$ so

$$v_1 = N_1 \frac{d\phi}{dt}$$

$$v_2 = N_2 \frac{d\phi}{dt}$$

And if the same flux is through both coils then

$$\frac{v_1}{v_2} = \frac{N_1}{N_2}$$

And if power is conserved:

$$v_1 i_1 = v_2 i_2$$

Or

$$\frac{i_1}{i_2} = \frac{v_2}{v_1} = \frac{N_2}{N_1}$$

(which can also be found from magnetic field arguments)

So the voltage change is proportional to the number of turns (higher N, higher V) and the current is inversely proportional (higher N, lower i).

So we can transform voltage, current, and voltage/current = impedance.

Note here we have assumed no winding or core loss, an ideal case.

Example

You would like $18V_{RMS}$ at 650mA from a $120V_{RMS}$ line voltage. You have a transformer with 1000 turn primary (input). How many turns are required in the secondary(output)? What is the primary current?

$$\frac{v_p}{v_s} = \frac{N_p}{N_s}$$

$$N_s = \frac{N_p v_s}{v_p} = \frac{1000 \cdot 18\sqrt{2}}{120\sqrt{2}} = 150 turns$$

(peak or rms values can be used).

And

$$i_P N_p = i_s N_s$$

So

$$i_P = \frac{650mA \cdot 150}{1000} = 97.5mA$$

Notice

$$v_s = \frac{N_s v_p}{N_p}$$

$$i_s = \frac{N_p i_p}{N_s}$$

$$v_p = \frac{N_p v_s}{N_s}$$

$$i_p = \frac{N_s i_s}{N_p}$$

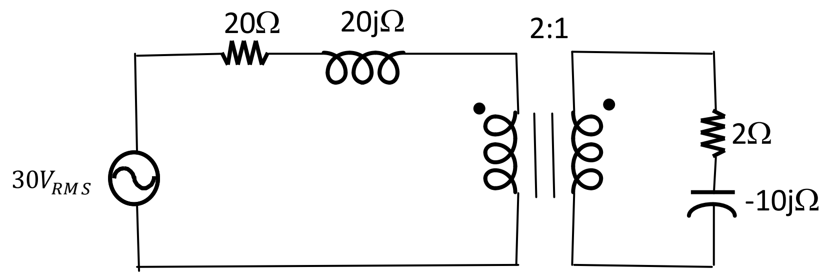
Therefore:

$$\frac{v_p}{i_p} = Z_p = \frac{\frac{N_p v_s}{N_s}}{\frac{N_s i_s}{N_p}} = \left(\frac{N_p}{N_s}\right)^2 \frac{v_s}{i_s}$$

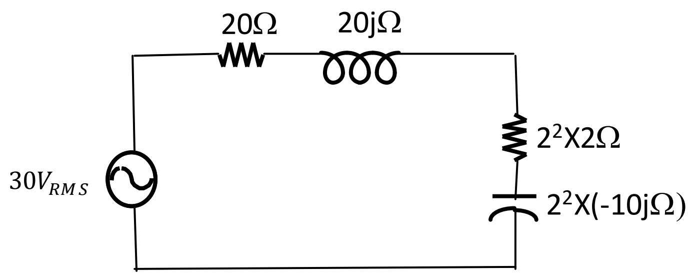
$$Z_p = \left(\frac{N_p}{N_s}\right)^2 Z_s$$

We can transform impedance as well!

Example: Find the current drawn from the source.



We first transform the impedance on the secondary to its equivalent resistance on the primary:



Now we find current in the usual way:

$$i_p = \frac{30\sqrt{2}\angle 0°}{20 + 20j + 8 - 40j} = 0.872\sqrt{2}\angle 35.5°A$$

Laplace Transforms

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Where s is complex frequency $s = \sigma + j\omega$

This goes from time domain -> complex frequency domain. Also the inverse:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}ds$$

Allows us to return to the time domain.

So this is very like our phasor analysis (D.E. -> algebraic) (used for cosine functions only) but is more general and captures sinusoidal S.S. ($s = j\omega$) and transient ($s = \sigma$) response and all combinations.

Hence you will find many transfer functions written in terms of s (s-domain complex freq domain).

You can solve many circuit problems without using the transform, so the mechanics and application can wait until you get more background in mathematics.

Fourier Analysis

Techniques apply to a wide range of engineering problems (heat flow, vibrations, circuits).

Any periodic function can be expressed as a sum of linearly independent functions such as sinusoidal functions.

$$f(t) = a_o + \sum_{n=1}^{\infty} D_n \cos(n\omega_o t + \theta_n)$$

i.e. a fundamental term (n=1) and all harmonics (multiples of ω_o)

and from Eulers identity

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_o t}$$

(an exponential Fourier series)

Pick one of the terms $e^{-jk\omega_o t}$, multiply and integrate over a period T_o:

$$\int_{t_1}^{t_1+T_o} f(t)e^{-jk\omega_o t}dt = \int_{t_1}^{t_1+T_o} \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_o t} \right) e^{-jk\omega_o t}dt$$

We have a bunch of terms $e^{j(n-k)\omega_o t}$ which average to zero, unless n=k.

$$= C_k T_o$$

So

$$C_k = \frac{1}{T_o} \int_{t_1}^{t_1+T_o} f(t)e^{-jk\omega_o t}dt$$

And given the function f(t) we can find coeff in the series.

So any periodic signal will have a frequency spectrum made up of the coefficients C_n (fundamental and harmonics).

And the response of a circuit to a periodic function is the superposition of the responses from each sinusoidal term:

$$v(t) = v_0 + v_1(t) + v_2(t) + \cdots$$

In fact, if the harmonic functions became close enough to each other (a continuous frequency spectrum) it can be shown that a periodic signal can also be represented

- ➔ Fourier Transform

Similar to Laplace, and again will probably be easier to use with more math background.