Poles and Zeroes

You may have noticed that our impedances take the form of a ratio of polynomials in $j\omega$ (or s)

$$Z(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

Although we have primarily seen 1st order functions (a_1, a_0, b_1, b_0).

This is also true for voltages and currents, admittances and gains.

In fact, there are 4 'gains' we may be interested in, depending on input and output variables

Input	Output	Transfer Function	
Voltage	Voltage	Voltage Gain	$G_v(s)$
Voltage	Current	Trans Admittance	$Y(s)$
Current	Current	Current Gain	$G_i(s)$
Current	Voltage	Trans Impedance	$Z(s)$

Simple Poles and Zeroes

A simple pole or zero has the form $(1 + j\omega\tau)$

Either a zero $1 + j\omega\tau \ \rightarrow \quad 0dB \quad \omega\tau \ll 1 \quad \phi = 0$

$\rightarrow \ +\frac{20dB}{dec} \quad \omega\tau \gg 1 \quad \phi = 90$

Break at $\tau = \frac{1}{\omega}$

Or a pole $\frac{1}{1+j\omega\tau} \ \rightarrow \quad 0dB \quad \omega\tau \ll 1 \quad \phi = 0$

$\rightarrow \ -\frac{20dB}{dec} \quad \omega\tau \gg 1 \quad \phi = -90$

Break at $\tau = \frac{1}{\omega}$

You may also have $(1 + j\omega\tau)^N$ which has slope and phase multiplied by N.

Quadratic Pole or Zero

Has the form $1 + 2\zeta(j\omega\tau) + (j\omega\tau)^2$ where ζ is the damping ratio.

This could be a RLC or resonant network.

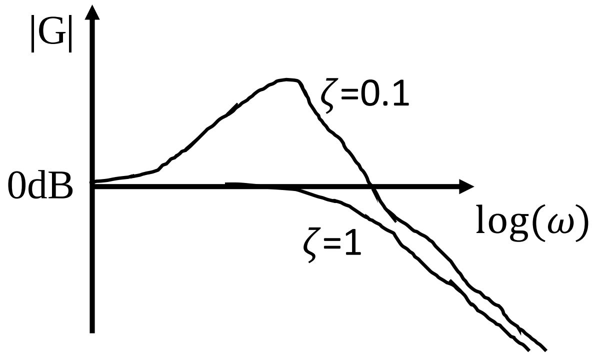
If $\zeta \geq 1$ roots are real and this is the product of two simple pole/zero terms.

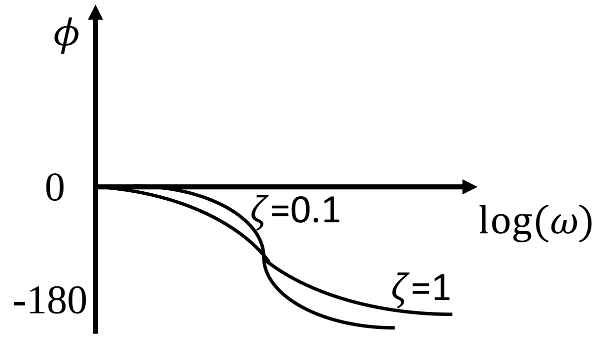
If $\zeta < 1$ roots are complex conjugates.

For $\omega\tau \ll 1$ magnitude -> 0dB $\omega\tau \gg 1$ magnitude $|(\omega\tau)^2| \rightarrow 40dB/dec$ +for zeros and – for poles.

In between the limits, behavior depends on ζ

Phase shift is $\tan^{-1}\frac{2\zeta\omega\tau}{1-(\omega\tau^2)}$ and goes from $0° \rightarrow\ \pm180°$ (positive for a zero and negative for a pole)





If you can factor a transfer function and recognize these terms, you can quickly plot the frequency dependent behavior without extensive calculation.

Example: Plot the following transfer function:

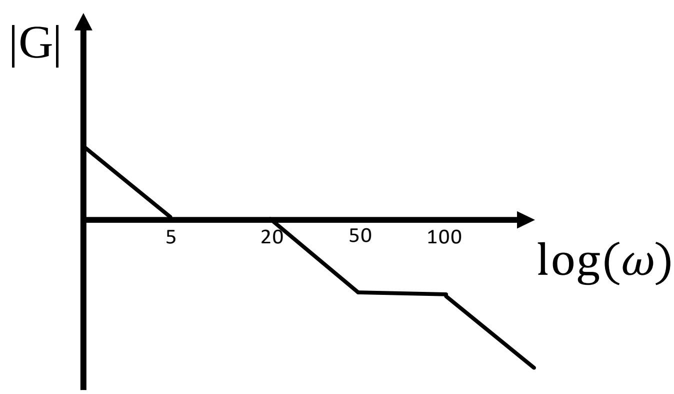
$$G(j\omega) = \frac{\left(1 + j\frac{\omega}{5}\right)\left(1 + j\frac{\omega}{50}\right)}{j\omega\left(1 + j\frac{\omega}{20}\right)\left(1 + j\frac{\omega}{100}\right)}$$

We note there are three poles at $\omega = 0, 20, 100$

And two zeros at $\omega = 5{,}50$

Below the pole/zero frequency the pole/zero has a magnitude of 1.

Above the pole/zero frequency a pole has a magnitude of $1/\omega$ and a zero has a magnitude of ω. Knowing this we can sketch:



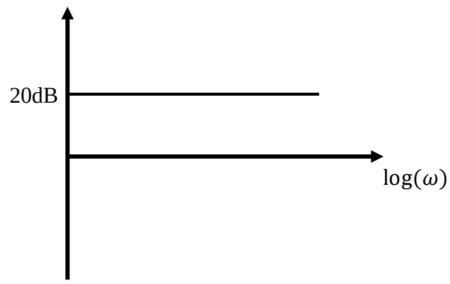
Example:

Plot the magnitude and phase as a function of frequency for:

$$G_V(j\omega) = \frac{10(1 + 0.1j\omega)}{(1 + j\omega)(1 + 0.02j\omega)}$$

Plot individual asymptotic plots for each term and add them since log(ab) = log(a)+log(b)

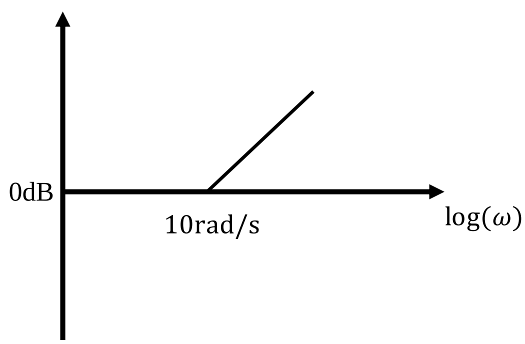
Term 1: $20\log(10) = 20dB = constant$



Term 2: $20\log|1 + 0.1j\omega| = 0dB \;\; 0.1\omega \ll 1$

$$= +\frac{20dB}{dec}\; 0.1\omega \gg 1$$

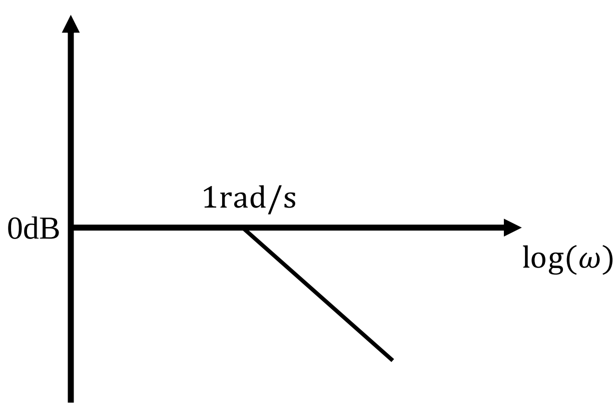
ω_c @ $0.1\omega = 1 \quad or\; \omega_c = \frac{10rad}{s}$



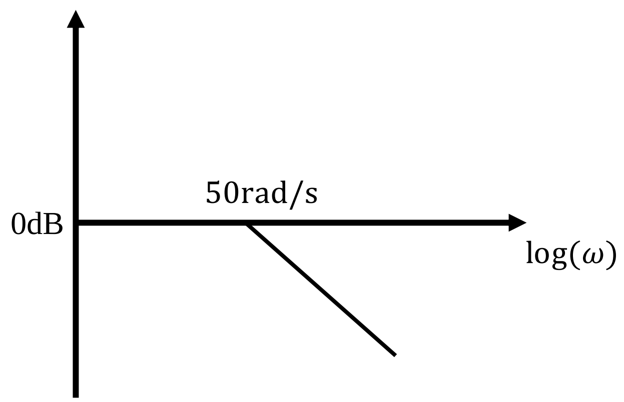
Term 3: $-20\log|1 + j\omega| = 0dB \quad \omega \ll 1$

$$= -\frac{20dB}{dec}\; \omega \gg 1$$

$\omega_c = \frac{1rad}{s}$



Similarly Term 4 gives:

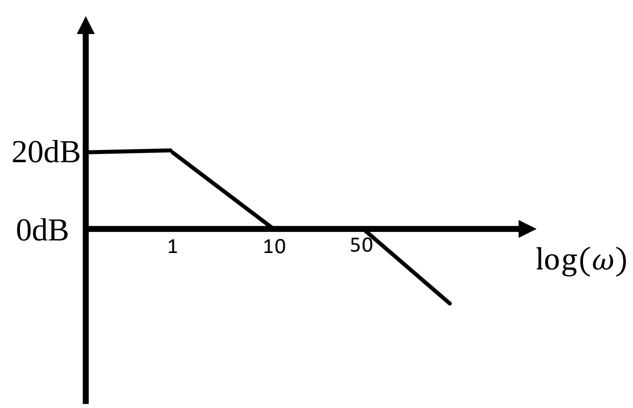


Composite magnitude is found by adding terms. E.g. at 1rad/s 20dB+0dB+0dB+0dB=20dB and magnitude begins changing at -20dB/decade.

At 10rad/s 20dB+0dB-20dB/dec +0dB = 0dB

And magnitude begins changing at -20dB/dec+20dB/dec = 0dB/dec.

At 50rad/s we are still at 0dB, but the magnitude begins changing at -20+20-20=-20dB/dec.



We can go through a similar process for phase

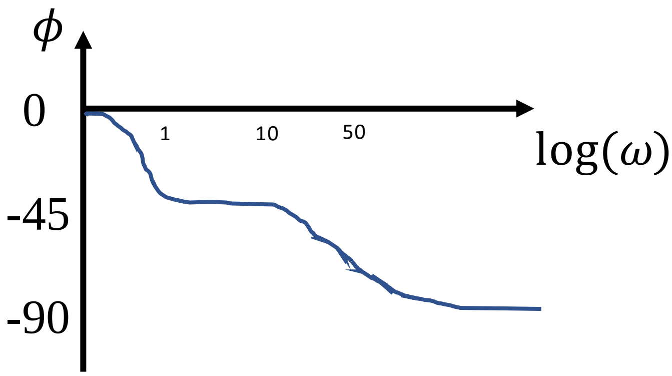
$K_0 = 10 \rightarrow \phi = 0°$ constant

$(1 + 0.1j\omega) \rightarrow \phi = 0° \quad \omega \ll 10 \quad \phi = 90° \quad \omega \gg 10$

$\frac{1}{1+j\omega} \rightarrow \phi = 0° \quad \omega \ll 1 \quad \phi = -90° \quad \omega \gg 1$

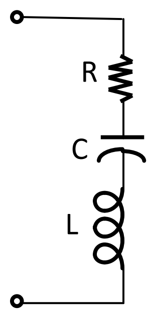
$\frac{1}{1+0.02j\omega} \rightarrow \phi = 0° \quad \omega \ll 50 \quad \phi = -90° \quad \omega \gg 50$

Summing these starts at 0°, begins dropping at $\omega = 1rad/s$, levels at $\omega = 10rad/s$and resumes its decrease to $-90°$ at $\omega = 50rad/s$.



Resonant Circuits

Series Resonance:



$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$= R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The imaginary term =0 if

$\omega L = \frac{1}{\omega C}$ or $\omega_o = \frac{1}{\sqrt{LC}}$ and $Z(j\omega) = R$

ω_o =resonant frequency.

Resonance is an important effect, sometimes avoided (e.g. vibrations) or exploited (tuned devices).

At resonance V and I are in phase.

Impedance is minimum, current is maximum.

At low frequencies, Capacitance dominates

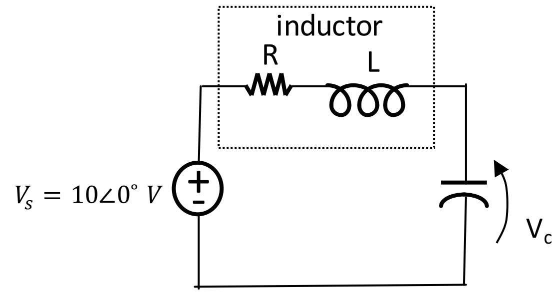
At high frequencies, inductance dominates.

Quality factor: $Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR} = \frac{1}{R}\sqrt{\frac{L}{C}}$

Q is an important characteristic of a resonant circuit -> low R = high Q

Example

Design a RLC network with resonance at 1000Hz using a 0.02H inductor with Q =200.



The required capacitor is:

$$\omega_o = \frac{1}{\sqrt{LC}} = 2\pi f_o$$

Or

$$C = \frac{1}{0.02H \cdot (2\pi \cdot 1000)^2} = 1.27\mu F$$

And from the inductor: $Q = \frac{\omega_o L}{R} = 200$

$$R = \frac{2\pi \cdot 1000 \cdot 0.02}{200} = 0.628\Omega$$

Then I = V_s/R=15.9A at resonance.

And the capacitor voltage would be

$$V_c = I \cdot \left(\frac{1}{j\omega C}\right) = 15.9 \cdot \left(\frac{125}{j}\right) \cong 200V!$$

So be careful of the voltage rating as there is lots of stored energy at resonance!

Impedance and Admittance of the resonant circuit is written in various forms. One convenient form is in terms of $\omega_o, \omega, and\ Q$.

$$Y(j\omega) = \frac{1}{Z(j\omega)} = \frac{1}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$= \frac{1}{R\left(1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right)}$$

and $Q = \frac{\omega_o L}{R}$ or $\frac{L}{R} = \frac{Q}{\omega_o}$

also $Q = \frac{1}{\omega_o CR}$ or $\frac{1}{RC} = Q\omega_o$

then

$$Y(j\omega) = \frac{1}{R + j\left(Q\frac{\omega}{\omega_o} - Q\frac{\omega_o}{\omega}\right)}$$

And since

$$I = \frac{V_s}{Z} = Y \cdot V_s$$

Where V_s is the source phasor and V_R = IR.

Then

$$\frac{V_R}{V_S} = \frac{I \cdot R}{V_S} = Y \cdot R$$

Or

$$G_V = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}$$

Which has a magnitude of

$$M(\omega) = \frac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2}}$$

And phase

$$\phi(\omega) = -\tan^{-1} Q\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)$$

To sketch these consider

$$\omega = \omega_o \qquad M(\omega) = 1 \ \ (0dB)$$

Half power frequencies give circuit bandwidth

$$\frac{1}{1 + Q^2\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2} = \frac{1}{2}$$

Or

$$Q\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) = \pm 1$$

And using the quadratic formula (after some rearrangement)

$$\omega_{LO} = \omega_o \left[\frac{-1}{2Q} + \sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \right]$$

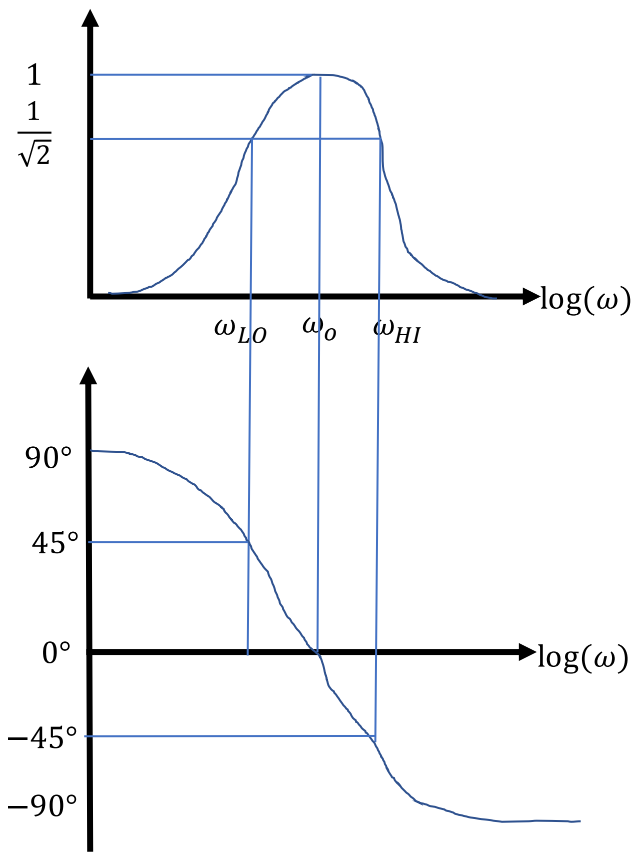
$$\omega_{HI} = \omega_o \left[\frac{1}{2Q} + \sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \right]$$

(note that negative frequencies are rejected). Which gives a difference

$$BW = \omega_{HI} - \omega_{LO} = 2 \cdot \frac{\omega_o}{2Q} = \frac{\omega_o}{Q}$$

And you can also show

$$\omega_{HI} \cdot \omega_{LO} = {\omega_o}^2$$



So small R = high Q = narrow bandwidth = high selectivity e.g. tuning or filtering.

Now consider energy storage in the resonant circuit.

At $\omega_o$the circuit impedance is R for source $V_m\angle 0°$ so

$$V_C = \frac{1}{j\omega_o C} I = \frac{1}{j\omega_o C}\frac{V_m}{R}\angle 0° = \frac{V_m}{\omega_o RC}\angle -90°$$

And energy stored

$$\omega_C(t) = \frac{1}{2}C{V_C}^2(t) = C \cdot \frac{{V_m}^2}{2{\omega_o}^2R^2C^2}\sin^2(\omega_o t)$$

For the inductor,

$$I = \frac{V_m}{R}\angle 0°$$

So

$$\omega_L(t) = \frac{1}{2}Li^2(t) = \frac{L}{2} \cdot \frac{{V_m}^2}{R^2}\cos^2(\omega_o t)$$

Since $\omega_o = \frac{1}{\sqrt{LC}}$ we can remove C from the capacitor stored energy, so

$$\omega_C(t) = \frac{{V_m}^2}{2\frac{1}{LC}R^2C}\sin^2(\omega_o t)$$

And total energy stored

$$\omega_S(t) = \omega_C(t) + \omega_L(t) = \frac{{V_m}^2L}{2R^2}(\sin^2(\omega_o t) + \cos^2(\omega_o t))$$

$$= \frac{{V_m}^2L}{2R^2}$$

(Since $\sin^2 + \cos^2 = 1$) is the maximum stored energy.

In resonance this energy is exchanged between capacitor and inductor (at freq ω_o)

And how much energy is dissipated in a cycle?

$$\omega_D = \int_0^T P_R dt = \int_0^T \left(\frac{V_m}{R}\cos(\omega_o t)\right)^2 R dt = \frac{{V_m}^2}{2R}T$$

The ratio of the stored energy to the dissipated energy is:

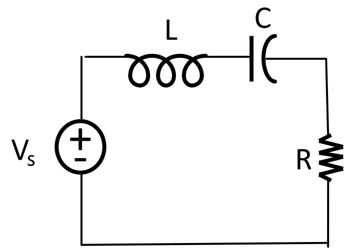
$$\frac{\omega_S}{\omega_D} = \frac{\frac{{V_m}^2L}{2R^2}}{\frac{{V_m}^2}{2R}T} = \frac{L}{RT} = \frac{L}{R\frac{2\pi}{\omega_o}} = \frac{\omega_o}{2\pi}\frac{L}{R} = \frac{Q}{2\pi}$$

So

$$Q = 2\pi\frac{\omega_S}{\omega_D}$$

A definition of Q is used in all resonant phenomena (mechanical, electrical, acoustic, etc.).

Example



Find R,L, C to give a BP filter with $\omega_o = 1000 rad/s$, BW = 100rad/s (usually in Hz)

$\omega_o = \frac{1}{\sqrt{LC}}$ so $\frac{1}{LC} = 10^6$

$BW = \frac{\omega_o}{Q}$ or $Q = \frac{1000}{100} = 10$

And

$Q = \frac{\omega_o L}{R}$ so $10 = \frac{1000L}{R}$

2 equations, 3 unknowns so fix C = 1μF, then L = 1H (a bit large) , R=100Ω.

We saw in an example that capacitor and inductor voltage in series resonant circuits can be high (Q X V_s).

Does the maximum actually occur at resonance (ω_o)?

$V_C = \left(\frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}\right) V_S$ (voltage division)

$= \frac{V_S}{1 - \omega^2 LC + j\omega RC}$

$$|V_C| = \frac{|V_S|}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

Now set

$$\frac{d|V_C|}{d\omega} = 0 = \frac{d}{d\omega}(1 - 2\omega^2 LC + \omega^2 R^2 C^2 + \omega^4 L^2 C^2)^{-1/2}$$

$$= -\frac{1}{2}((1 - \omega^2 LC)^2 + (\omega RC)^2)^{-3/2} \cdot (-4\omega LC + 2\omega R^2 C^2 + 4\omega^3 L^2 C^2)$$

(chain rule)

Or

$$2LC - R^2 C^2 = 2\omega^2 L^2 C^2$$

$$\omega^2 = \frac{1}{LC} - \frac{1}{2}\frac{R^2}{L^2}$$

$$\omega_{max} = \sqrt{\frac{1}{LC} - \frac{1}{2}\frac{R^2}{L^2}}$$

$$\omega_{max} = \sqrt{\omega_o{}^2 - \frac{1}{2}\left(\frac{\omega_o}{Q}\right)^2}$$

$$= \omega_o\sqrt{1 - \frac{1}{2Q^2}}$$

So clearly $\omega_{max} \neq \omega_o$ but approaches it as Q gets large.

Now back to the capacitor voltage:

$$|V_C| = \frac{|V_S|}{\sqrt{(1 - \omega_{max}{}^2 LC)^2 + (\omega_{max} RC)^2}}$$

$$= \frac{|V_S|}{\sqrt{\left(1 - \left(\omega_o{}^2 - \frac{1}{2}\frac{\omega_o{}^2}{Q^2}\right)\cdot\frac{1}{\omega_o{}^2}\right)^2 + \left(\omega_o{}^2 - \frac{\omega_o{}^2}{2Q^2}\right) R^2C^2}}$$

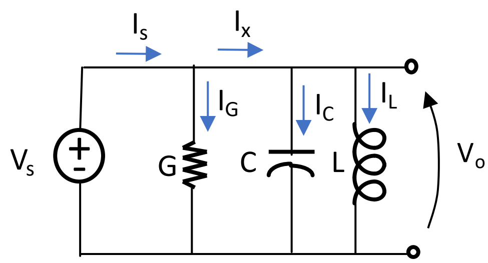
$$= \frac{|V_S|}{\sqrt{\left(-\frac{1}{2Q^2}\right)^2 + \frac{1}{Q^2}\left(1 - \frac{1}{2Q^2}\right)}}$$

$$= \frac{|V_S|}{\sqrt{\frac{1}{Q^2}\left(\frac{1}{4Q^2} + \left(1 - \frac{1}{2Q^2}\right)\right)}}$$

$$= \frac{Q|V_S|}{\sqrt{1 - \frac{1}{4Q^2}}} \sim Q|V_S|$$

For large Q.

Parallel Resonance



In this case $I_s = Y \cdot V_s$

$$I_s = V_s\left(G + j\left(\omega C - \frac{1}{\omega L}\right)\right)$$

And at $\omega = \omega_o$ $I_s = G \cdot V_s$

There are still currents I_c and I_L but they are 180° out of phase, and I_X -> 0.

Energy is transferred from inductor to capacitor and back again.

Admittance is determined by inductance at low frequencies and capacitance at high frequencies.

$$Q = \frac{R}{\omega_o L} = \frac{1}{G\omega_o L} = \omega_o RC = \frac{\omega_o C}{G}$$

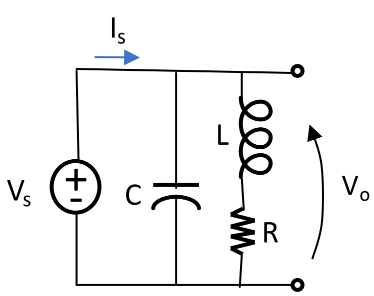
Which you can derive from $Q = \frac{\omega_S}{\omega_D}$ but is analogous to the series case $\frac{L}{R} \to RC$ or $Q_S \to \frac{1}{Q_P}$

Currents mimic voltage in series case. e.g.

$$|I_c| = Q|I_s|$$

Notice that V_o/V_s =1 always in this case and we would be more interested in say V_o/I_s a <u>transimpedance</u> characteristic.

Since inductor windings has resistance, other circuit variations occur:



$$Y(j\omega) = j\omega C + \frac{1}{R + j\omega L}$$

$$= j\omega C + \frac{R - j\omega L}{R^2 + \omega^2 L^2}$$

$$Y(j\omega) = \frac{R}{R^2 + \omega^2 L^2} + j\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)$$

And the resonance condition (Y real)

$$\omega_r C = \frac{\omega_r L}{R^2 + \omega_r{}^2 L^2}$$

$$R^2 + \omega_r{}^2 L^2 = \frac{L}{C}$$

$$\omega_r{}^2 = \frac{1}{LC} - \left(\frac{R}{L}\right)^2$$

$$\omega_r = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

And if R is small it is quite similar to a pure parallel resonant circuit. Referring back to our quadratic pole/zero term

$$1 - (\omega\tau)^2 + 2j\zeta\omega\tau$$

Where $\tau = \frac{1}{\omega_o}$ and our series impedance

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C} = \frac{R \cdot j\omega C + (j\omega)^2 LC + 1}{j\omega C}$$

$$= \frac{1}{j\omega C}(1 - \omega^2 LC + j\omega RC)$$

Then by comparison of terms:

$$\tau^2 \to LC = \frac{1}{\omega_o{}^2}$$

$$2\zeta\tau \to RC$$

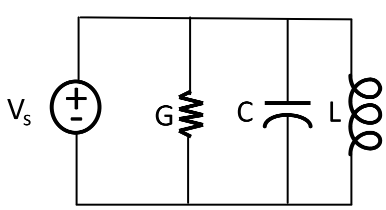
$$\zeta = \frac{RC}{2\tau} = \frac{RC\omega_o}{2} = \frac{1}{2Q}$$

Or

$$Q = \frac{1}{2\zeta}$$

High damping -> low Q, low damping -> high Q.

Example



$V_s = 120\angle 0°$, G = 0.01S, C = 600μF, L = 120mH. Find the branch currents at resonance. First:

$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{120 \cdot 10^{-3} \cdot 600 \cdot 10^{-6}}} = 117.85 rad/s$$

Required to find admittance/impedance:

$$Y_c = j\omega_o C = 7.07 \cdot 10^{-2} jS$$

$$Y_L = \frac{1}{j\omega_o L} = -7.07 \cdot 10^{-2} jS$$

Note same magnitude! And:

$$I_G = GV_S = 1.2\angle 0°$$

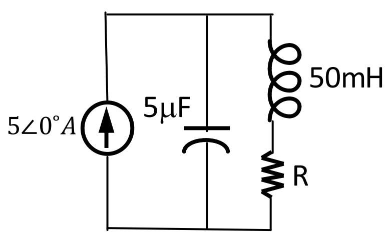
$$I_C = Y_c V_S = 8.49\angle 90°$$

$$I_L = Y_L V_S = 8.49\angle -90°$$

So

$$I_S = I_G + I_C + I_L = 1.2\angle 0° = I_G$$

<u>Example</u>



Find ω_o and ω_r for R = 50Ω and 5Ω.

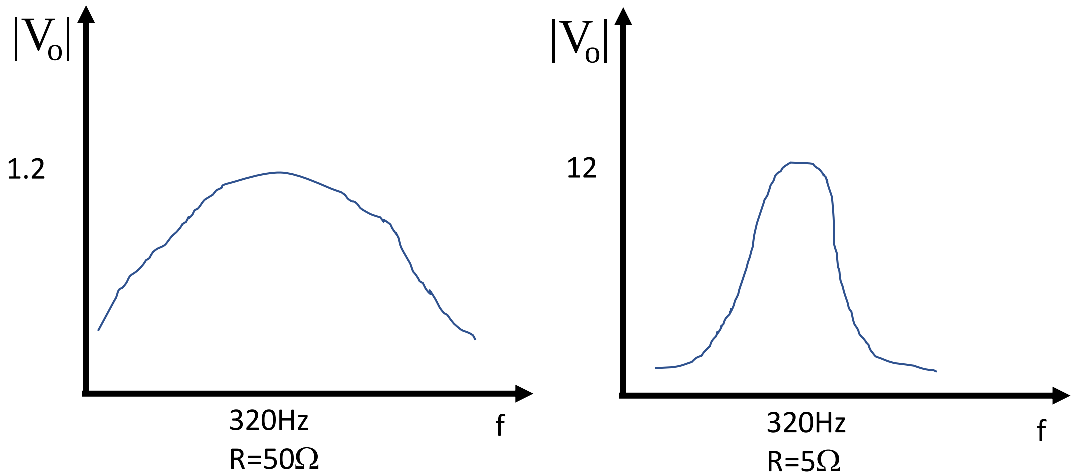
$$\omega_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{50 \cdot 10^{-3} \cdot 5 \cdot 10^{-6}}} = \frac{2000 rad}{s} = 318.3 Hz$$

If R = 50Ω

$$\omega_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \sqrt{\frac{1}{50 \cdot 10^{-3} \cdot 5 \cdot 10^{-6}} - \left(\frac{50}{50 \cdot 10^{-3}}\right)^2} = \frac{1732 rad}{s} = 275.7 Hz$$

If R = 5Ω

$$\omega_r = \sqrt{\frac{1}{50 \cdot 10^{-3} \cdot 5 \cdot 10^{-6}} - \left(\frac{5}{50 \cdot 10^{-3}}\right)^2} = \frac{1997 rad}{s} = 317.9 Hz$$


|Vo|
1.2
320Hz
f
R=50Ω
|Vo|
12
320Hz
f
R=5Ω