Phasor Analysis with Kirchoff's Laws

Kirchoff's laws apply in the frequency domain and can be used to analyze steady state AC circuit response.

Voltages and currents must be expressed as phasors, then the analysis is similar to resistive circuits but with complex coefficients.

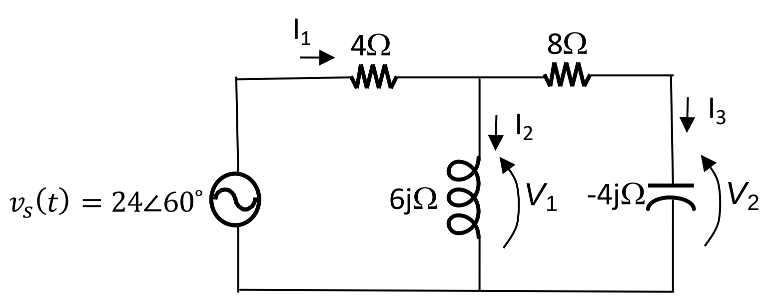
For simple circuits:

- Ohm's Law   V = I Z
- Combinations Z_s, Y_P
- KVL, KCL
- Current and voltage division

For more complicated circuits:

- Nodal analysis
- Loop analysis
- Superposition
- Thevenin and Norton's theorems

Example Find all Voltages and Currents:



We can use the equivalent impedance Z_{eq} to find current I_1:

$$Z_{eq} = 4 + 6j//(8-4j)$$

$$= 4 + \frac{6j(8-4j)}{6j+8-4j}$$

$$= 4 + \frac{24+48j}{8+2j}$$

$$= 4 + \frac{53.66\angle 63.43}{8.246\angle 14.04}$$

$$= 4 + 6.51\angle 49.39$$

$$= 4 + 4.24 + 4.94j = 8.24 + 4.94j$$

$$Z_{eq} = 9.61\angle 30.94$$

Then

$$I_1 = \frac{V}{Z_{eq}} = \frac{24\angle 60}{9.61\angle 30.94} = 2.5\angle 29.06$$

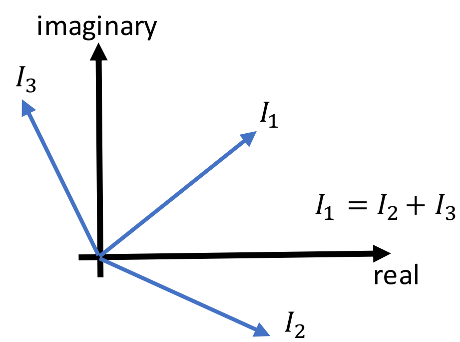
And

$$V_1 = v_s - 4I_1$$
$$= 24\angle 60 - 4 \cdot 2.5\angle 29.06$$
$$= (12 + 20.78j) - (8.47 + 4.86j)$$
$$= 3.26 + 15.92j$$
$$= 16.25\angle 78.4$$

Voltage division would also work:

$$V_1 = v_s \cdot \frac{6j//(8-4j)}{4 + 6j//(8-4j)}$$
$$= v_s \cdot \frac{6.51\angle 49.39}{9.61\angle 30.94}$$
$$= \frac{24 \cdot 6.51}{9.61}\angle(60 + 49.39 - 30.94)$$

$$= 16.26\angle 78.45$$



Knowing V_1 we can find I_2 and I_3:

$$I_2 = \frac{V_1}{6j} = \frac{16.26\angle 78.45}{6\angle 90} = 2.71\angle -11.58$$

$$I_3 = \frac{V_1}{8-4j} = \frac{16.26\angle 78.45}{8.94\angle -26.56} = 1.82\angle 105$$

Current division also works:

$$I_2 = I_1 \cdot \frac{8-4j}{8-4j+6j} = \frac{(2.5\angle 29.06)\cdot(8.94\angle -26.56)}{8.24\angle 14.04} = 2.71\angle -11.54$$

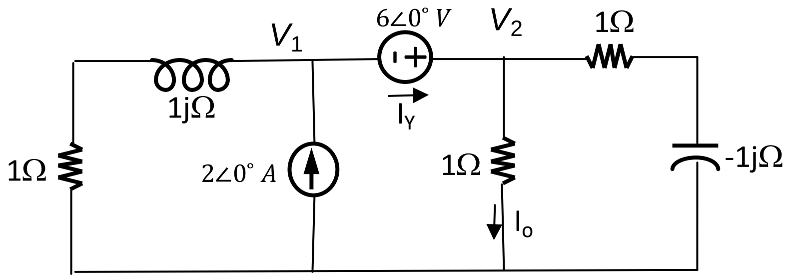
Finally

$$V_2 = I_3 \cdot -4j = 1.82 \cdot 4\angle(105-90) = 7.28\angle 15$$

Analysis Techniques

Phasor techniques can be applied to more complicated circuits

Example find I_o:



We first note that:

$$V_2 = V_1 + 6\angle 0$$

(there is a voltage source connecting these two nodes)

1) KCL solution:

Node 1:

$$\frac{V_1}{1+j} - 2\angle 0 + I_Y = 0$$

$$I_Y = 2\angle 0 - \frac{V_1}{1+j}$$

Node 2:

$$\frac{V_2}{1} + \frac{V_2}{1-j} - I_Y = 0$$

$$I_Y = \frac{V_2}{1} + \frac{V_2}{1-j}$$

$$I_Y = V_1 + 6\angle 0 + \frac{V_1 + 6\angle 0}{1-j}$$

Setting these two equations to be equal (remove I_Y):

$$V_1 + 6\angle 0 + \frac{V_1 + 6\angle 0}{1-j} = 2\angle 0 - \frac{V_1}{1+j}$$

$$V_1 + \frac{V_1}{1-j} + \frac{V_1}{1+j} = -4\angle 0 - \frac{6\angle 0}{1-j}$$

$$V_1\left(1 + \frac{1}{1-j} + \frac{1}{1+j}\right) = -4 - \frac{6}{1-j}$$

$$V_1\left(1 - j + 1 + \frac{1-j}{1+j}\right) = -4 + 4j - 6$$

$$V_1\left(2 - j + \frac{-2j}{2}\right) = -10 + 4j$$

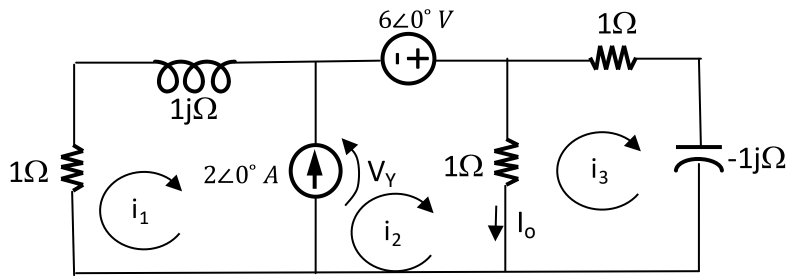
$$V_1(2-2j) = -10+4j$$

$$V_1 = \frac{-10+4j}{2-2j} = \frac{10.77\angle(158.2)}{2\sqrt{2}\angle(-45)} = 3.81\angle 203.2$$

Now $V_2 = V_1 + 6\angle 0 = -3.50 - 1.5j + 6 = 2.5 - 1.5j = 2.92\angle -31$

Therefore: $I_o = \frac{V_2}{1} = 2.92\angle -31A$

2) KVL Solution:



We first note that:

$$i_2 - i_1 = 2\angle 0$$

$$i_2 = i_1 + 2$$

Loop1:

$$-i_1(1+j) - V_Y = 0$$

$$V_Y = -i_1(1+j)$$

Loop 2:

$$V_Y + 6\angle 0 - 1(i_2 - i_3) = 0$$

$$V_Y = (i_2 - i_3) - 6$$

$$V_Y = (i_1 + 2 - i_3) - 6$$

$$V_Y = i_1 - i_3 - 4$$

Loop 3:

$$1(i_2 - i_3) - i_3(1-j) = 0$$

$$i_2 - 2i_3 + i_3 j = 0$$

$$i_1 + 2 - 2i_3 + i_3 j = 0$$

$$i_1 = 2i_3 - i_3 j - 2$$

Removing V_Y from the first two loop equations yields:

$$-i_1(1+j) = i_1 - i_3 - 4$$

$$-i_1 - i_1 j = i_1 - i_3 - 4$$

$$i_3 = 2i_1 + ji_1 - 4$$

Using the third loop equation to remove i_1:

$$i_3 = 2(2i_3 - i_3 j - 2) + j(2i_3 - i_3 j - 2) - 4$$

$$i_3 = 4i_3 - 2i_3 j - 4 + 2ji_3 + i_3 - 2j - 4$$

$$i_3 = 5i_3 - 8 - 2j$$

$$i_3 = 5i_3 - 8 - 2j$$

$$4i_3 = 8 + 2j$$

$$i_3 = 2 + 0.5j$$

Therefore:

$$i_1 = \ 2i_3 - i_3 j - 2$$

$$i_1 = 4 + j - 2j + 0.5 - 2$$

$$i_1 = 2.5 - j$$

And

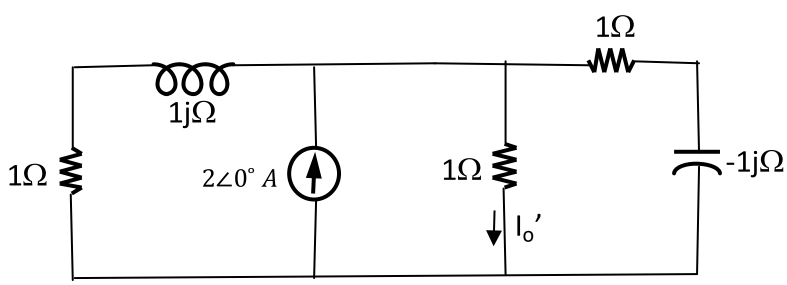
$$i_2 = i_1 + 2 = 4.5 - j$$

Finally:

$$I_o = i_2 - i_3 = 4.5 - j - 2 - 0.5j = 2.5 - 1.5j = 2.92\angle - 31$$

3) Superposition Solution:

Apply one independent source at a time starting with the current source:

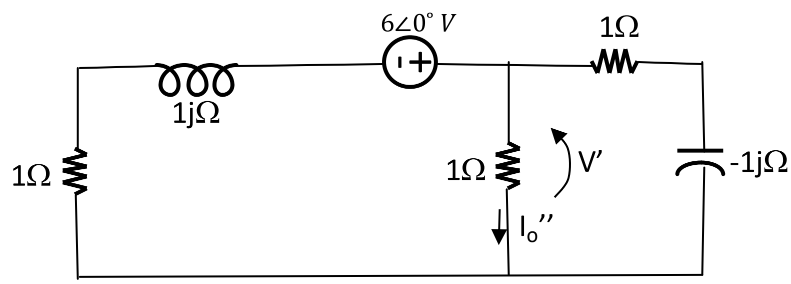


Combine the impedances in the branches we are not interested in:

$$Z' = (1 + j)//(1 - j) = \frac{(1 + j)(1 - j)}{(1 + j) + (1 - j)} = \frac{2}{2} = 1$$

Thus we have simply two 1ohm resistors in parallel so each one will draw half the current and I_o'=1A

Now the voltage source:



A voltage divider exists between 1+j on the left and 1//(1-j) on the right so:

$$(1-j)//1 = \left(1+\frac{1}{1-j}\right)^{-1}$$

$$= \left(\frac{2-j}{1-j}\right)^{-1} = \frac{1-j}{2-j} = \frac{1-j}{2-j}\left(\frac{2+j}{2+j}\right) = \frac{2+j-2j+1}{4+1} = \frac{3-j}{5} = 0.6-0.2j$$

And knowing that:

$$V' = 6\frac{0.6-0.2j}{(0.6-0.2j)+(1+j)} = 6\frac{0.6-0.2j}{1.6+0.8j} = 6\frac{0.632\angle -18.43}{1.79\angle 26.56} = 2.11\angle -45$$

And therefore:

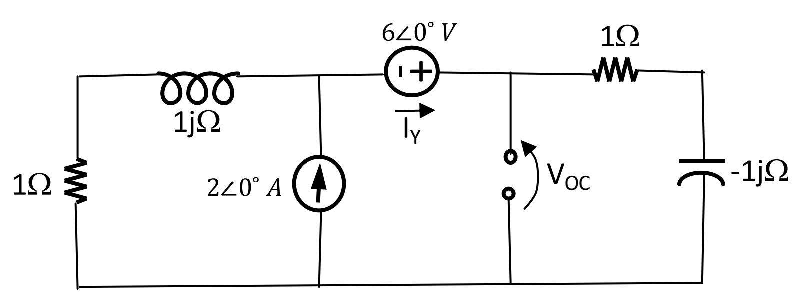
$$I_o'' = \frac{V'}{1} = 2.11\angle -45$$

So the total current is:

$$I_o = I_o' + I_o'' = 1 + 2.11\angle -45 = 1 + 1.5 - 1.5j = 2.5 - 1.5j = 2.91\angle -31$$

Which is the same as the last two solutions (strange????).

4) We could use source exchange (transformation) not that commonly used…
5) Thevenin Analysis



We have removed the 1ohm load and now we will use superposition.

$$V_{OC} = 2A\cdot\frac{1+j}{(1+j)+(1-j)}\cdot(1-j) + 6V\cdot\frac{1-j}{1+j+1-j}$$

(note the first term is superposition on the current source and the second term is superposition on the voltage source).

$$= 2A \cdot \frac{1+j}{2} \cdot (1-j) + \frac{6V}{2} \cdot (1-j)$$

$$= 2A \cdot 1 + (3-3j) = (5-3j)V$$

And:

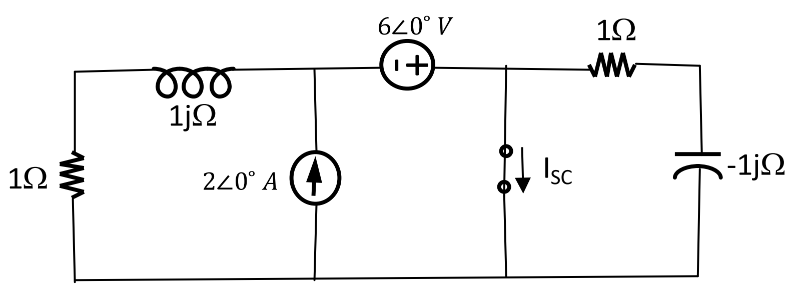
$$Z_{TH} = (1+j)//(1-j)$$

$$= \frac{(1+j)(1-j)}{1+j+1-j} = \frac{1}{2} \cdot 2 = 1\Omega$$

Reconnecting the load:

$$I_O = \frac{V_{TH}}{2} = 2.91\angle -31$$

6) Norton Analysis



$$I_{SC} = 2 + \frac{6}{1+j} = \frac{2+2j+6}{1+j} = \frac{8+2j}{1+j}$$

Z_{th} is the same as before and is 1 ohm. Since the load is also 1ohm:

$$I_o = \frac{1}{2} I_{SC} = \frac{4+j}{1+j} = \frac{(4+j)(1-j)}{2} = 2.5 - 1.5j = 2.91\angle -31$$

Filter Networks

We know enough now to design simple passive filters.

Filters are circuits designed to pass signals in a specific frequency range, and attenuate or reject signals outside this range.

Low-pass filters pass low frequencies and reject high frequencies.

High-pass filters pass high frequencies and reject low frequencies.

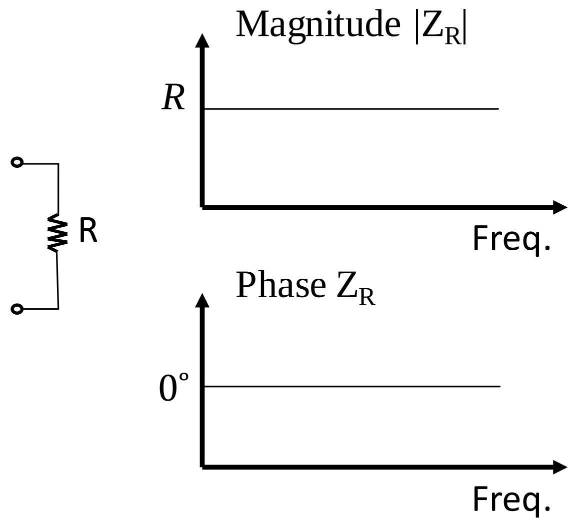
Band-pass filters pass a frequency band.

Band-reject filters are designed to reject a specific frequency band.

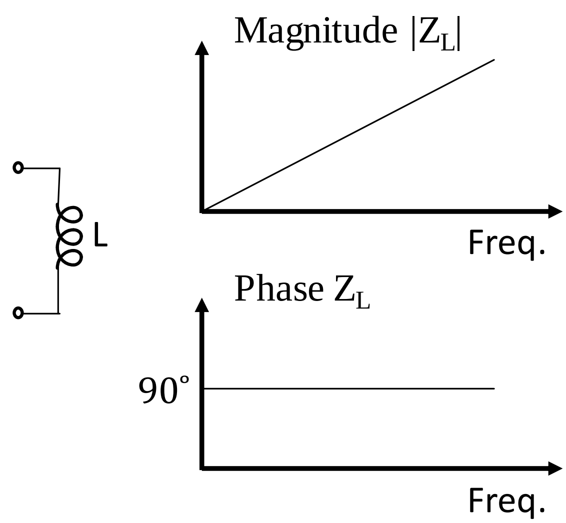
An ideal filter would have a definite cut-off frequency ω_o completely passing all signals on one side, rejecting all signals on the other side. However, a real filter will have a more gradual cut-off.

Consider element impedance as a function of frequency:

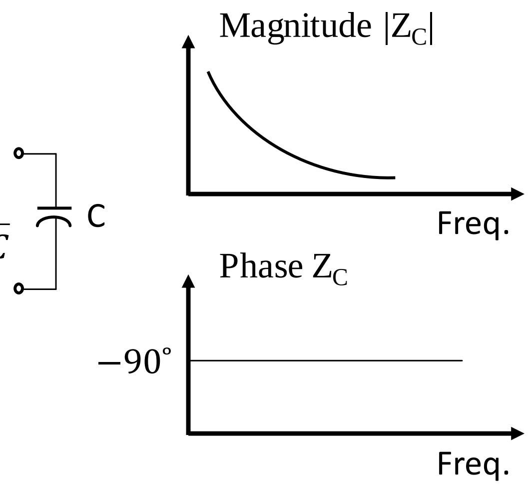
$Z_R = R$



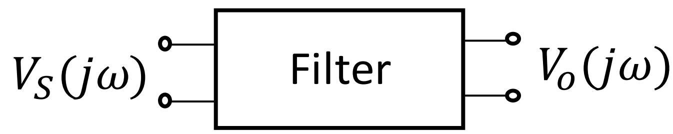
$Z_L = j\omega L$



$$Z_C = \frac{1}{j\omega C}$$



For a filter we are interested in a ratio of output to input



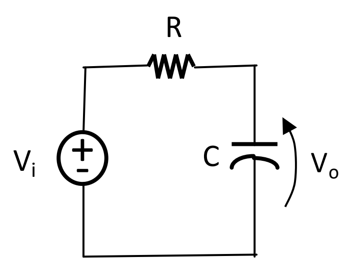
And we use a gain as a transfer function to express this:

$$G_V(j\omega) = \frac{V_o(j\omega)}{V_s(j\omega)}$$

Often we use $s = j\omega$ (for a Laplace transform, complex freq)

$$G_V(s) = \frac{V_o(s)}{V_s(s)}$$

Consider a simple RC circuit:



From voltage divider:

$$V_o = V_i \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$G(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 + j\omega RC}$$

$$G(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 + j\omega\tau}$$

$\tau = RC$

Magnitude:

$$|G(\omega)| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

Phase:

$$\phi(\omega) = -tan^{-1}(\omega\tau)$$

Maximum occurs when $\omega = 0$ use define a break frequency when $|G(\omega)| = \frac{1}{\sqrt{2}}$corresponding to ½ maximum power (V^2/Z) or:

$$\omega = \frac{1}{\tau}$$

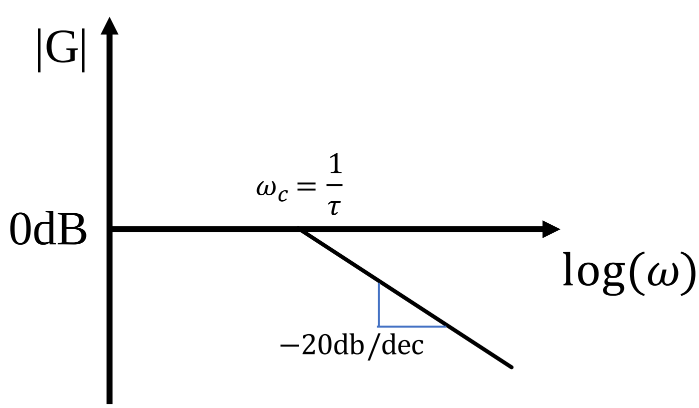
Magnitude is expressed in decibels = dB = 20log|G|. We can plot this approximately using 3 points:

$$\omega \to 0 \quad |G| = 1 \ = 0dB$$

$$\omega = \frac{1}{\tau} \quad |G| = \frac{1}{\sqrt{2}} \ = -3dB$$

$$\omega \to \infty \quad |G| \cong \frac{1}{\omega\tau} \ = -20\log(\omega) - 20\log(\tau)\, dB$$

$\left(slope\ of\ \frac{-20dB}{decade}\right)$



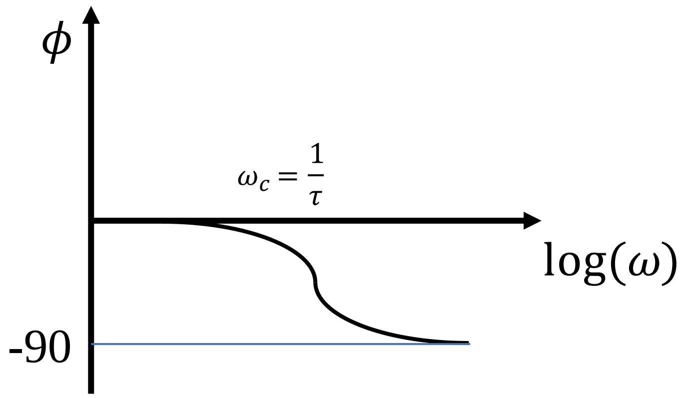
This is a bode plot.

We can use a similar approach for phase:

$$\omega \to 0 \quad \phi \cong 0$$

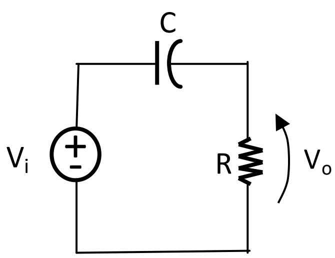
$$\omega = \frac{1}{\tau} \quad \phi = -45$$

$$\omega \to \infty \quad \phi \cong -90$$



This would be a passive low pass filter.

Now consider voltage across the resistor in an RC network:



$$V_o = V_i \frac{R}{R + \frac{1}{j\omega C}}$$

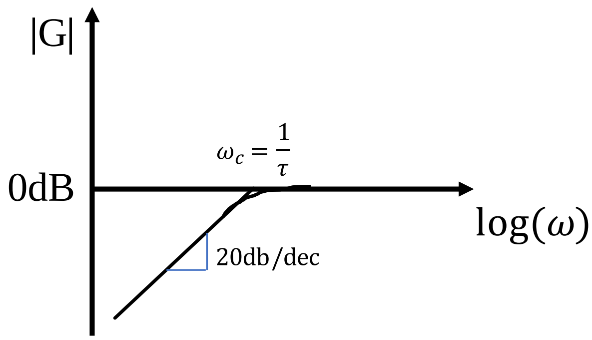
$$G(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{j\omega RC}{1 + j\omega RC}$$

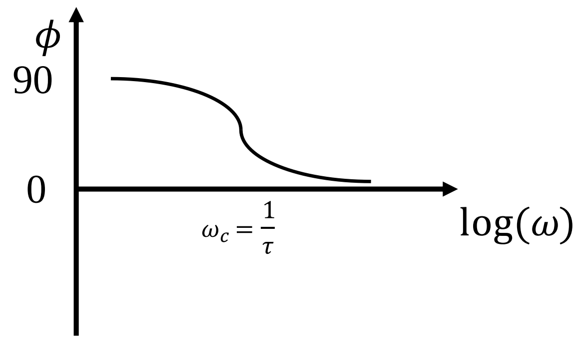
Consider the same three regions:

$$\omega \to 0 \quad |G| \cong \omega\tau \; = 20\log(\omega) + 20\log(\tau)$$

$$\omega = \frac{1}{\tau} \quad |G| = \frac{1}{\sqrt{2}} \; = -3dB$$

$$\omega \to \infty \quad |G| = \frac{\omega\tau}{\omega\tau} \; = 1 = 0dB$$





$$\omega \to 0 \quad \phi = 90$$

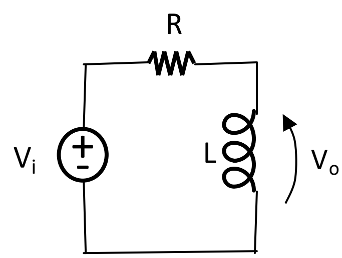
$$\omega = \frac{1}{\tau} \quad \phi = 45$$

$$\omega \to \infty \quad \phi = 0$$

This is called a high-pass characteristic.

$$\phi(\omega) = 90 - tan^{-1}(\omega\tau)$$

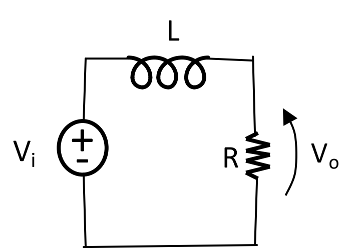
A series RL circuit could also be used:



$$V_o = V_i \frac{j\omega L}{R + j\omega L}$$

$$G(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R}} = \frac{j\omega\tau}{1 + j\omega\tau}$$

This is a high pass filter.

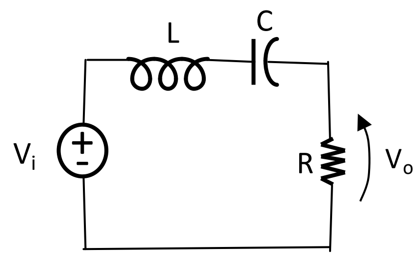


$$V_o = V_i \frac{R}{R + j\omega L}$$

$$G(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 + j\omega\frac{L}{R}} = \frac{1}{1 + j\omega\tau}$$

This is a low pass filter. So these look like the RC examples (but $\tau = \frac{L}{R}$)

What if we have three elements?



$$V_o = V_i \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

$$G(j\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

At low frequencies $\omega \to 0$

$$|G(j\omega)| \cong \omega RC \cong 0$$

Slope is 20dB/dec.

At high frequencies $\omega \to \infty$

$$|G(j\omega)| \cong \frac{R}{\omega L} \cong 0$$

Slope is -20dB/dec.

When $\omega L = \frac{1}{\omega C}$

$$|G(j\omega)| = 1 = 0dB$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

And 3dB (1/2 power) points are when:

$R = \pm\left(\omega L - \frac{1}{\omega C}\right)$

(There are two points).

$$\begin{cases} \omega CR = \omega^2 LC - 1 \\ \omega CR = 1 - \omega^2 LC \end{cases}$$

$$\begin{cases} \omega^2 + \frac{R}{L}\omega - \frac{1}{LC} = 0 \\ \omega^2 - \frac{R}{L}\omega - \frac{1}{LC} = 0 \end{cases}$$

Noting that $\omega_o = \frac{1}{\sqrt{LC}}$

$$\begin{cases} \omega^2 + \frac{R}{L}\omega - {\omega_o}^2 = 0 \\ \omega^2 - \frac{R}{L}\omega - {\omega_o}^2 = 0 \end{cases}$$

Solving gives the upper and lower -3dB points:

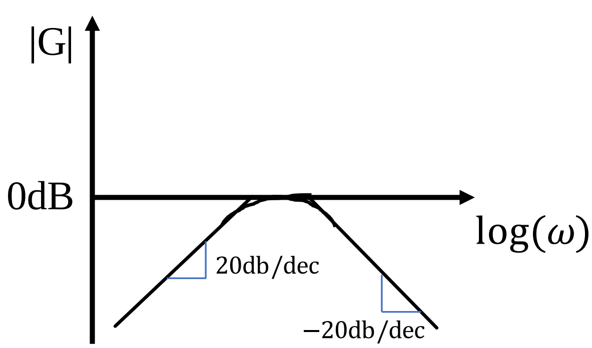
$$\omega_{LO} = \frac{-\frac{R}{L} + \sqrt{\left(\frac{R}{L}\right)^2 + 4{\omega_o}^2}}{2}$$

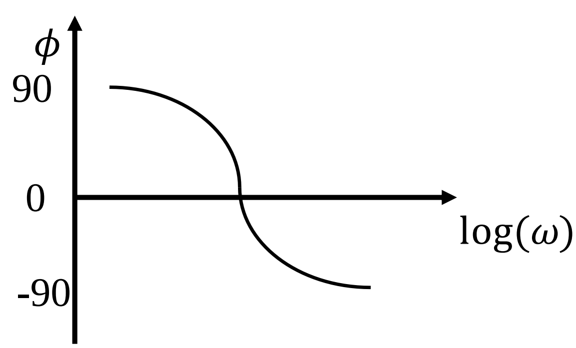
$$\omega_{HI} = \frac{\frac{R}{L} + \sqrt{\left(\frac{R}{L}\right)^2 + 4{\omega_o}^2}}{2}$$

The filter bandwidth is:

$$\omega_{HI} - \omega_{LO} = \frac{R}{L}$$

This is a bandpass filter.





And a band-reject is possible taking the voltage across the LC: