AC Steady-State Analysis – Phasors and Frequency Domain Analysis

At the start of the course we discussed the sine wave

$$x(t) = X_m \sin(\omega t)$$

X_m = Amplitude or peak value

ω = angular frequency (radians/sec)

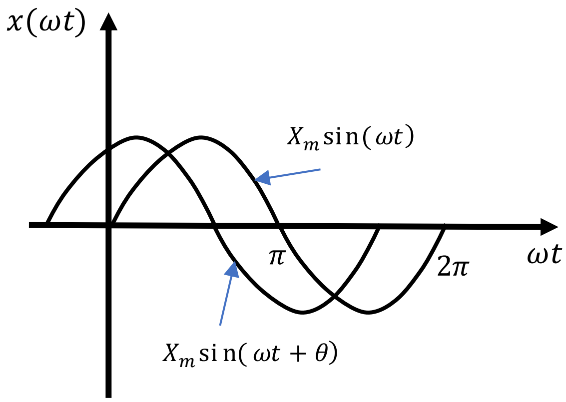
Obviously, the function repeats every 2π radians. So:

$$x(t+T) = x(t)$$

For period $T = \frac{1}{f} = \frac{2\pi}{\omega}$

And we may also have a phase angle measured from the same reference.

$$x(t) = X_m \sin(\omega t + \theta)$$



Note $X_m \sin(\omega t + \theta)$ <u>leads</u> $X_m \sin(\omega t)$ since it peaks at an earlier time. We could also say $X_m \sin(\omega t)$ <u>lags</u> $X_m \sin(\omega t + \theta)$

Two sinusoids with the same phase angle are 'in phase'. Phase angle is usually given in degrees rather than radians. We may write

$$x(t) = X_m \sin(\omega t + \frac{\pi}{2})$$

Or

$$x(t) = X_m \sin(\omega t + 90^\circ)$$

ωt is in radians, so this isn't rigorously correct, but it generally easy to understand. Notice that sine and cosine are functions that differ only by phase angle

$$cos(\omega t) = \sin(\omega t + \frac{\pi}{2})$$

Or

$$sin(\omega t) = cos\ (\omega t - \frac{\pi}{2})$$

And we can use either form depending on our phase reference.

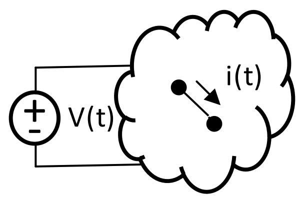
To compare phase of sinusoids we need them to:

- ➔ Have the same frequency
- ➔ Have positive amplitude (negative = 180° phase shift)
- ➔ Both should be sine wave or cosine waves (90°phase difference)

You may want to review some sinusoid identities, in particular sum and difference of angles, half or double angle formulas, $\sin^2$, $\cos^2$.

Sinusoidal Forcing Functions

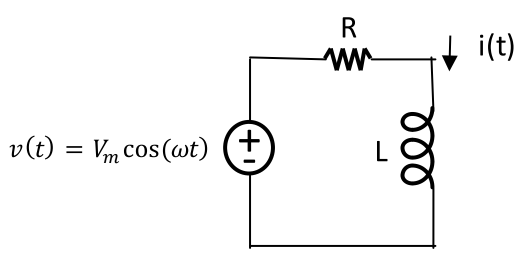
If we apply a sinusoid to a linear circuit, it will produce sinusoidal steady state voltage and current (at the same frequency as the input).



If $v(t) = A\sin\ (\omega t + \theta)$ then i(t) must have the form $i(t) = B\sin\ (\omega t + \phi)$ and to find a solution we need only find B and ϕ.

Example:

Say we have a simple RL circuit:



As with transient response, KVL gives us:

$$L\frac{di(t)}{dt} + Ri(t) = V_m cos(\omega t)$$

We can assume a solution for the forced response

$i(t) = Acos\ (\omega t + \phi)$ (*******)

$= Acos(\phi)\cos\ (\omega t) - Asin(\phi)\sin\ (\omega t)$

$= A_1 \cos(\omega t) + A_2 \sin\ (\omega t)$

We can substitute this into our D.E.

$$L(-A_1\omega\sin(\omega t) + A_2\omega\cos(\omega t)) + R(A_1\cos(\omega t) + A_2\sin(\omega t)) = V_m cos(\omega t)$$

Collect terms in sin and cos:

$$-A_1\omega L + RA_2 = 0$$

$$A_2\omega L + RA_1 = V_m$$

2 equations for A_1 and A_2. Solving these:

$$A_1 = \frac{RV_m}{R^2 + \omega^2 L^2} = Acos(\phi)$$

$$A_2 = \frac{\omega L V_m}{R^2 + \omega^2 L^2} = -Asin(\phi)$$

The last terms above are from (******)

So $\tan(\phi) = \frac{Asin(\phi)}{Acos(\phi)} = \frac{\omega L}{R}$

And $(Acos(\phi))^2 + (Asin(\phi))^2 = A^2 = \frac{{V_m}^2}{R^2+\omega^2 L^2}$

So $i(t) = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} cos(\omega t - tan^{-1}(\frac{\omega L}{R}))$

Which is a rather tedious approach for a simple circuit….

We can simplify analysis by using Euler's equation:

$$e^{j\omega t} = cos(\omega t) + jsin(\omega t)$$

Where j represents a complex number $\sqrt{-1}$ (often i in other disciplines. We like i for current, so use j).

Real part: $Re(e^{j\omega t}) = cos(\omega t)$

Imaginary part: $Im(e^{j\omega t}) = sin(\omega t)$

Now say we have an imaginary forcing function $v(t) = V_m e^{j\omega t}$ (not realizable)

Then $v(t) = V_m cos(\omega t) + jV_m sin(\omega t)$ from Euler's equation.

So the complex forcing function has two parts, and from linearity and superposition we could expect a current:

$i(t) = I_m cos(\omega t) + jI_m sin(\omega t) = I_m e^{j(\omega t+\phi)}$

So we could analyze circuit response to a function $Ae^{j(\omega t)}$and extract response to $A\cos(\omega t)$why? -> properties of the exponential allow us to convert the problem to an algebraic one.

Lets return to the RL circuit, but instead of applying $V_m cos(\omega t)$, we use $V_m e^{j\omega t}$. Forced response will be:

$$i(t) = I_m e^{j(\omega t+\phi)}$$

With I_m and ϕ unknown. The D.E. is:

$$L\frac{d}{dt}\left(I_m e^{j(\omega t+\phi)}\right) + RI_m e^{j(\omega t+\phi)} = V_m e^{j(\omega t)}$$

And taking the derivative

$$j\omega LI_m e^{j(\omega t+\phi)} + RI_m e^{j(\omega t+\phi)} = V_m e^{j\omega t}$$

$e^{j\omega t}$appears in each term so:

$$j\omega LI_m e^{j(\phi)} + RI_m e^{j(\phi)} = V_m$$

And:

$$I_m e^{j(\phi)} = \frac{V_m}{R + j\omega L} = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} e^{-j\left(tan^{-1}\left(\frac{\omega L}{R}\right)\right)}$$

(convert to polar form see appendix)

So $I_m = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}}$ and $\phi = -tan^{-1}\left(\frac{\omega L}{R}\right)$

However, our actual forcing function was $V_m cos(\omega t)$, so the circuit response is the real part of this solution:

$$i(t) = I_m cos(\omega t + \phi) = \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} cos\left(\omega t - tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

Which is identical to the previous result.

Notice the expression $e^{j\omega t}$was in every term cancelled out. So we can write:

$$v(t) = V_m cos(\omega t + \theta) = Re\left[V_m e^{j(\omega t+\theta)}\right]$$

$$= Re\left[V_m e^{j\theta} e^{j\omega t}\right]$$

$$= Re\left[V_m \angle\theta \; e^{j\omega t}\right]$$

And $e^{j\omega t}$ appears in every term of our equations, so we don't need to write it all the time, and in fact we will usually just write:

$V = V_m\angle\theta$ or $I = I_m\angle\phi$

'phasor notation' -> magnitude and phase only.

Phase angle is based on cosine function so:

$$Acos(\omega t \pm \theta) \rightarrow A\angle \pm \theta$$

$$Asin(\omega t \pm \theta) \rightarrow A\angle \pm \theta - 90^\circ$$

So for our RL series circuit:

$$L\frac{di(t)}{dt} + Ri(t) = V_m cos(\omega t)$$

Becomes:

$$L\frac{d}{dt}\left(Ie^{j\omega t}\right) + RIe^{j\omega t} = Ve^{j\omega t}$$

Or:

$$j\omega LI + RI = V \qquad (*)$$

Then

$$I = \frac{V}{R + j\omega L} = I_m\angle\phi = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}\angle - tan^{-1}\left(\frac{\omega L}{R}\right)$$

And

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} cos\left(\omega t - tan^{-1}\left(\frac{\omega L}{R}\right)\right)$$

Note that phasors are usually expressed with positive magnitudes.

Analysis in equation (*) above, where time dependence is not included is called phasor or frequency domain analysis.

Differential equations + sinusoidal forcing function in the time domain.

Algebraic equations with complex numbers in frequency domain.

Circuit Elements

Resistor:

$$v(t) = Ri(t)$$

$$V_m e^{j(\omega t + \theta_v)} = RI_m e^{j(\omega t + \theta_i)}$$

$$V_m e^{j\theta_v} = RI_m e^{j\theta_i}$$

$$V = RI$$

R is real and doesn't change phase between v and i.

Inductor:

$$v(t) = L\frac{di}{dt}$$

$$V_m e^{j(\omega t + \theta_v)} = L\frac{d}{dt}\left[I_m e^{j(\omega t + \theta_i)}\right]$$

$$V_m e^{j(\theta_v)} = j\omega LI_m e^{j(\theta_i)}$$

$$V = j\omega LI$$

$j\omega L$is complex and introduces a 90° phase shift between V and I. Voltage leads current by 90°.

Capacitor:

$$i(t) = C\frac{dv(t)}{dt}$$

$$I_m e^{j(\omega t+\theta_i)} = C\frac{d}{dt}\left[V_m e^{j(\omega t+\theta_v)}\right]$$

$$I = j\omega CV$$

$j\omega C$ is complex and introduces a 90° phase shift between V and I. Current leads current by 90°.

Example:

Voltage $v(t) = 24cos(377t + 75^{\circ})V$ is applied across a 6Ω resistor. What is the current?

Phasor Voltage $V = 24\angle 75^{\circ} \quad V$

Phasor Current $I = \frac{V}{R} = \frac{24\angle 75^{\circ}}{6} = \ 4\angle 75^{\circ}\, A$

Or time domain current:

$$i(t) = 4cos(377t + 75^{\circ})\ A$$

Example:

Voltage $v(t) = 12cos(377t + 20^{\circ})V$ is applied across a 20mH inductor. What is the current?

$$I = \frac{V}{j\omega L} = \frac{12\angle 20^{\circ}}{\omega L\angle 90^{\circ}} = \frac{12\angle 20^{\circ}}{377(20 \cdot 10^{3})\angle 90^{\circ}} = 1.59\angle(20 - 90)^{\circ} = 1.59\angle(-70)^{\circ}\, A$$

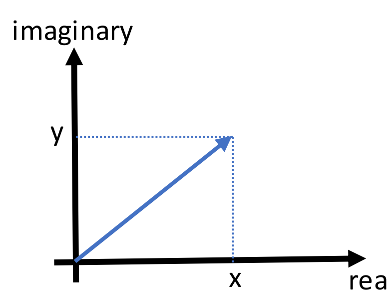
Time domain:

$$i(t) = 1.59cos(377t - 70^{\circ})\ A$$

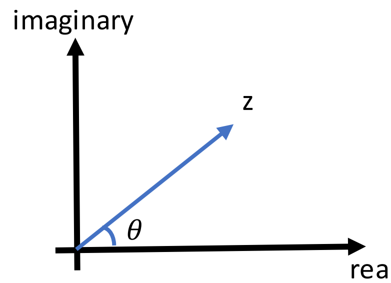
Complex Numbers

Rectangular representation

$$A = x + jy$$



Can also be written in polar coordinates



$$A = z\angle\theta$$

And

$z = \sqrt{x^2 + y^2}$

$\theta = tan^{-1}\left(\frac{y}{x}\right)$

$x = zcos(\theta)$

$y = zsin(\theta)$

Sums are easy in rectangular form:

$$A = 4 + j3 = 5\angle 36.9^\circ$$

$$B = 3 + j4 = 5\angle 53.1^\circ$$

$$A + B = (4 + 3) + j(3 + 4) = 7 + j7 = 9.9\angle 45^\circ$$

Products are easy in polar form

$$A \cdot B = 5 \cdot 5\angle(36.9 + 53.1)^\circ = 25\angle 90^\circ$$

Note polar form: $Z_1 e^{j\theta_1} \cdot Z_2 e^{j\theta_2} = Z_1 \cdot Z_2 e^{j\theta_1} e^{j\theta_2} = Z_1 Z_2 e^{j(\theta_1+\theta_2)}$

<u>**Impedance and Admittance**</u>

Notice for each element, the phasor approach gave us an algebraic equation for the I-V relationship,

$V = I \cdot Z$ where $Z = \begin{cases} R \\ j\omega L \\ \frac{1}{j\omega C} \end{cases}$ depending on the element.

We can generalize and define two terminal input <u>impedance</u> or driving point impedance

$$Z = \frac{V}{I} = \frac{V_m}{I_m}\angle(\theta_v + \theta_i) \quad [\Omega]$$

In rectangular form

$$Z(\omega) = R(\omega) + jX(\omega)$$

R = real (resistive) part

X = imaginary (reactive) part

Note Z depends on frequency (due to reactance), Z is complex, but is not a phasor since it is not a sinusoid.

For a Resistor $Z = R$

Inductor: $Z = j\omega L = jX_L = \omega L\angle 90^\circ$

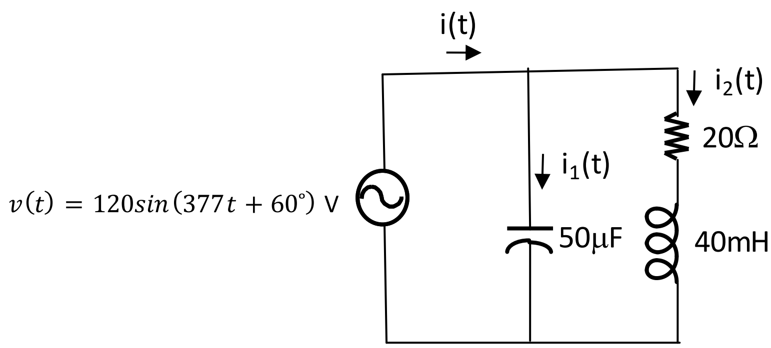
Capacitor: $Z = \frac{1}{j\omega C} = jX_C = \frac{1}{\omega C}\angle -90^\circ = -\frac{1}{\omega C}\angle 90^\circ$

KVL and KCL are valid in the frequency domain. You can go back to analysis for series and parallel resistance to show:

$$Z_S = Z_1 + Z_2 + \cdots + Z_N$$

$$\frac{1}{Z_P} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}$$

Example Find i(t):



$$\omega L = 377 \cdot 40 \cdot 10^{-3} = 15.08$$

$$\frac{1}{\omega C} = 53.05$$

Phasor voltage (as cosine) $V = 120\angle(60 - 90)^\circ = 120\angle(-30)^\circ$

Find i_2(t) impedance:

$$Z_2 = R + j\omega L = 20 + j15.08 = 25\angle(37)^\circ$$

$$I_2 = \frac{V}{Z_2} = \frac{120\angle(-30)^\circ}{25\angle(37)^\circ} = 4.8\angle(-67)^\circ\ A$$

Find i_1(t) impedance:

$$I_1 = \frac{V}{Z_1} = \frac{120\angle(-30)^\circ}{-j53.05} = \frac{120\angle(-30)^\circ}{53.05\angle(-90)^\circ} = 2.26\angle 60^\circ\ A$$

Find total current:

$$I = I_1 + I_2 = 4.8\angle(-67)^\circ + 2.26\angle(60)^\circ$$

$$= (1.87 - j4.42) + (1.13 + j1.957)$$

$$= 3 - j2.46 = 3.88\angle(-39.35)^\circ$$

Or we can combine impedances:

$$Z = (R + j\omega L)//\left(\frac{1}{j\omega C}\right)$$

$$= (20 + j15)//-j53$$

$$= \left(\frac{(20 + j15)(-j53)}{20 + j15 - j53}\right)$$

$$= \left(\frac{(53X15) - j1060}{70 - j38}\right) = \frac{1325\angle(-53.13)^{\circ}}{42.94\angle(-62.24)^{\circ}}$$

$$= 30.86\angle(-53.13 + 62.24)^{\circ}$$

$$= 30.86\angle(9.11)^{\circ}$$

And

$$I = \frac{V}{Z} = \frac{120\angle(-30)^{\circ}}{30.86\angle(9.11)^{\circ}} = 3.89\angle(-39.1)^{\circ}$$

Or

$$i(t) = 3.89cos(377t - 39.1^{\circ})\ A$$

Conductance

Recall conductance $G = \frac{1}{R}$

There is a similar quantity associated with impedance which is called admittance

$$Y = \frac{1}{Z} = \frac{I}{V} \quad [S]$$

$$Y = G + jB$$

G = conductance

B = susceptance

And

$$G + jB = \frac{1}{R + jX}$$

Then $G = \frac{R}{R^2 + X^2}$ and $B = \frac{-X}{R^2 + X^2}$

And using KVL and KCL you can show:

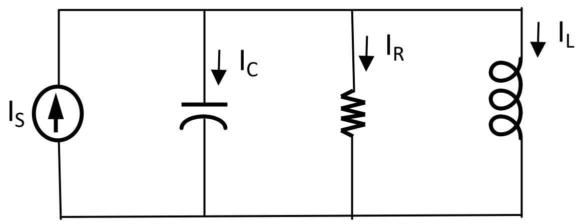
$$Y_P = Y_1 + Y_2 + \cdots + Y_N$$

$$\frac{1}{Y_S} = \frac{1}{Y_1} + \frac{1}{Y_2} + \cdots + \frac{1}{Y_N}$$

So we are most likely to use admittance in parallel circuits and in series mainly focus on impedance.

Sometimes circuit operation and relationships between parameters are most easily seen with a graphical representation -> a phasor diagram shows currents and voltages in the complex plane.

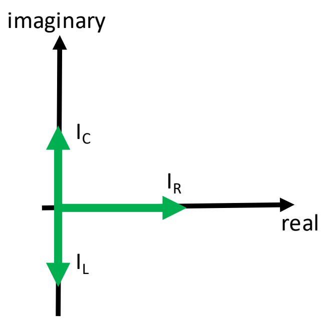
Example:



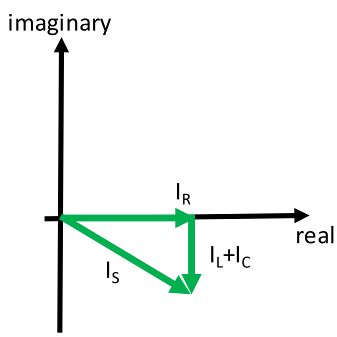
$$I_S = I_R + I_C + I_L = \frac{V}{R} + \frac{V}{j\omega L} + \frac{V}{\frac{1}{j\omega C}}$$

If $V = V_m\angle 0^\circ$ (an arbitrary phase reference) then:

$$I_S = \frac{V_m}{R}\angle 0^\circ + \frac{V_m}{\omega L}\angle - 90^\circ + V_m\omega C\angle 90^\circ$$

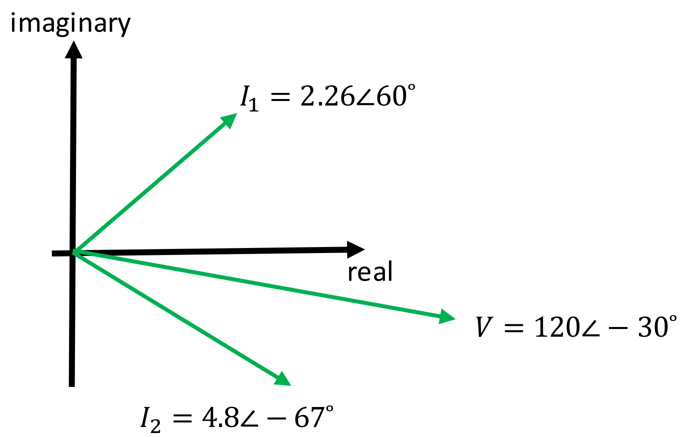


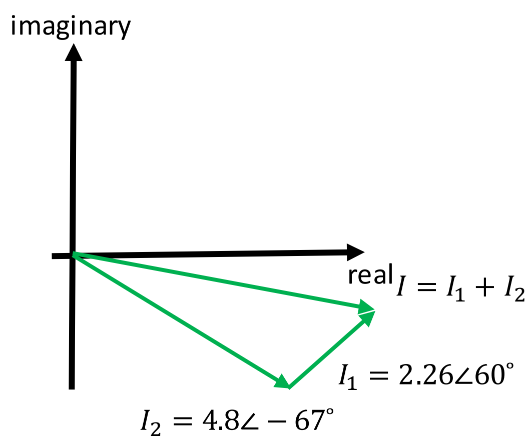
I_S can be found by summing the three currents life vectors.



Note at some frequency $|I_L| = |I_C|$ or $\frac{1}{\omega L} = \omega C \rightarrow \omega = \sqrt{\frac{1}{LC}}$

Of course voltage and current can have any relative phase. Lets consider our last numerical example which we just solved:

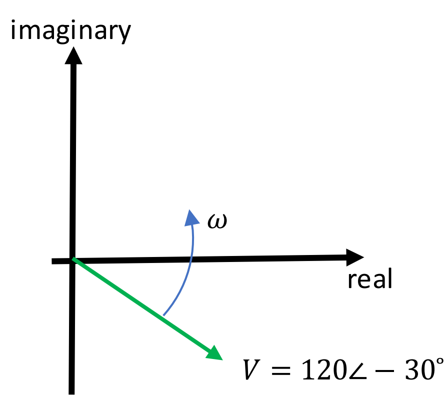




Note you can only observe the real part of voltage or current at any time -> but it changes predictably with time.

When we do phasor analysis we work at some instant in time, and our solution applies to all times.

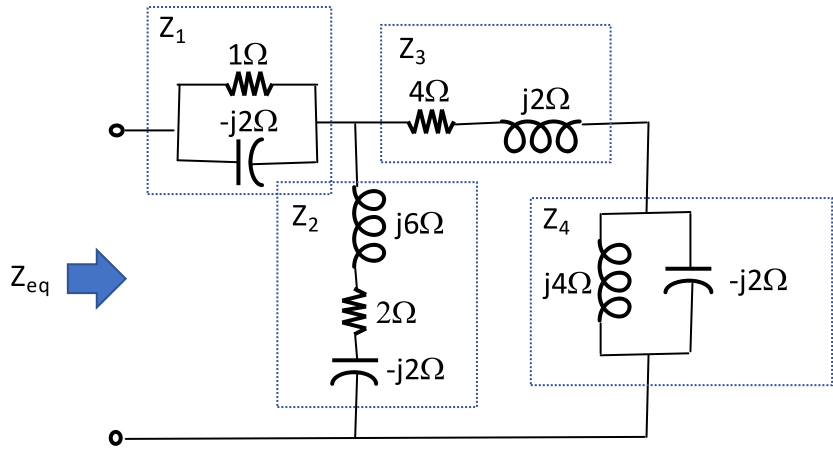
It might help you to picture the phasors rotating counter clockwise in time with angular frequency ω.



As the phasor rotates, it sweeps out a sinusoidal pattern on the real axis.

All the phasor quantities rotate together, the phase between them doesn't change (if frequency is fixed).

Example Find Z_{eq}:



Z_2 and Z_3 are easily found:

Z_2 = 2+j4 Z_3= 4+j2

Now using admittance:

$$Y_1 = Y_R + Y_C = \frac{1}{1} + \frac{1}{-j2} = 1 + 0.5j$$

$$Z_1 = \frac{1}{Y_1} = \frac{1}{1 + 0.5j} = 0.8 - 0.4j$$

$$Y_4 = Y_L + Y_C = \frac{1}{4j} + \frac{1}{-2j} = 0.25j$$

$$Z_4 = \frac{1}{Y_4} = -4j$$

Now we note that Z_3 and Z_4 are in series:

$$Z_{34} = Z_3 + Z_4 = 4 + 2j - 4j = 4 - 2j$$

Z_{34} is in parallel with Z_2 so:

$$Y_{234} = Y_2 + Y_{34} = \frac{1}{2 + 4j} + \frac{1}{4 - 2j} = 0.3 - 0.1j$$

$$Z_{234} = 3 + j$$

This is in series with Z_1 so:

$$Z_{eq} = Z_1 + Z_{234} = 0.8 - 0.4j + 3 + j = 3.8 + 0.6j$$