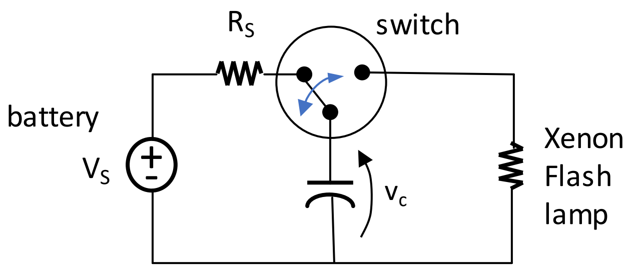
First and Second Order Transient Circuits

When conditions in a circuit change, for example when you turn it on or switch operating states, its behavior is transient. Circuit parameters change over some time allowing the change of state, then approach steady state operation.

Note periodic signals are changing over time, but since conditions repeat at regular intervals, the circuit reaches a periodic (AC) steady state.

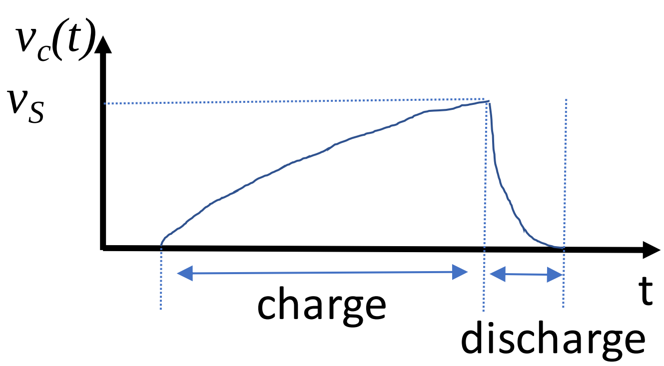
At this stage, to analyze transients we will work in the time domain -> with equations including t or derivatives d/dt.

Consider a camera flash operating from a battery:

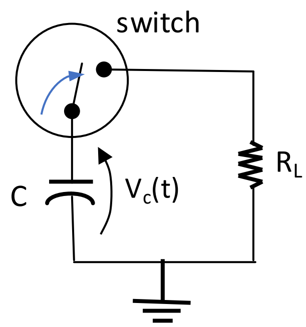


The battery charges a capacitor (fairly slowly). When the switch is moved (depressed) the capacitor discharges rapidly through the lamp.

We may be interested in the charging and discharging process:



Consider the discharge circuit:



When the switch is closed, KCL at a node between R and C:

$$C\frac{dv_c(t)}{dt}+\frac{v_c(t)}{R_L}=0$$

$$\frac{dv_c(t)}{dt}+\frac{1}{CR_L}v_c(t)=0$$

Which has a solution

$$v_c(t)=v_o e^{\frac{-t}{R_L C}}$$

Exponential charging with a rate that depends on the RC product.

<u>First Order Transients</u>

In general, the problem will have the form:

$$\frac{dx(t)}{dt}+ax(t)=f(t)$$

And the solution can be a linear combination of forced response (including f(t)) and natural response (with f(t) = 0) so:

$x(t)=x_p(t)+x_c(t)$ (particular and complementary)

If we limit our cases to a constant for the forcing function (A), then a solution can also be a constant K_1:

$$\frac{dx_p(t)}{dt}+ax_p(t)=A$$

$$x_p(t)=K_1$$

For the homogeneous equation (=0)

$$\frac{dx_c(t)}{dt}+ax_c(t)=0$$

$$\frac{1}{x_c(t)}\frac{dx_c(t)}{dt}=-a$$

You may recognize a solution

$$\ln(x_c(t)) = -at + c$$

Or

$$x_c(t) = K_2 e^{-at}$$

And we can put together a general solution:

$$x(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$$

Where K_1 is the steady state solution and τ is the time constant.

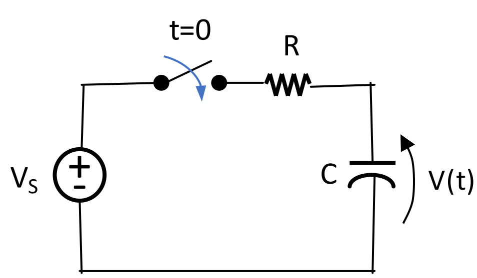
And we can solve problems by fitting our initial and final conditions or by writing and solving D.E.s.

Transient Circuit Analysis

-differential equation approach

-> write equation for voltage on a capacitor or current through an inductor.

Note: These quantities don't change instantaneously.



Write KCL:

$$C\frac{dv(t)}{dt} + \frac{v(t) - V_S}{R} = 0$$

Or

$$\frac{dv(t)}{dt} + \frac{v(t)}{RC} = \frac{V_S}{RC}$$

Which should have a solution

$$v(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$$

Substitute this into the differential equation:

$$\left(\frac{-K_2}{\tau} e^{\frac{-t}{\tau}}\right) + \frac{1}{RC}\left(K_1 + K_2 e^{\frac{-t}{\tau}}\right) = \frac{V_S}{RC}$$

Noting that the constant terms must be equal:

$$\frac{K_1}{RC} = \frac{V_S}{RC}$$

Or $K_1 = V_S$

The exponential terms must also be equal so:

$$\frac{K_2}{\tau} = \frac{K_2}{RC}$$

Or $\tau = RC$

We still need K_2 -> consider the initial condition

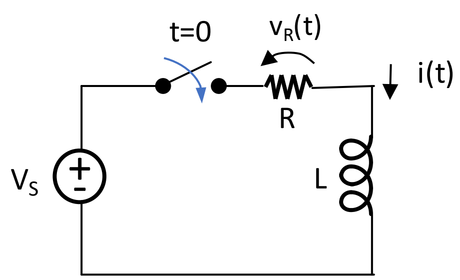
$V(0) = 0V = K_1 + K_2 = V_s + K_2$

$K_2 = -V_s$

And

$$v(t) = V_S - V_S e^{\frac{-t}{RC}} = V_S\left(1 - e^{\frac{-t}{RC}}\right)$$

If we had an inductor instead



Write KVL:

$$L\frac{di(t)}{dt} + Ri(t) = V_S$$

And

$$i(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$$

$$\left(\frac{-K_2}{\tau} e^{\frac{-t}{\tau}}\right) + \frac{R}{L}\left(K_1 + K_2 e^{\frac{-t}{\tau}}\right) = \frac{V_S}{L}$$

Constant terms:

$$\frac{K_1 R}{L} = \frac{V_S}{L}$$

$$K_1 = \frac{V_S}{R}$$

Exponential terms:

$$\frac{K_2}{\tau} = \frac{K_2 R}{L}$$

$$\tau = \frac{L}{R}$$

And at t =0

$$i(0) = 0 = \frac{V_S}{R} + K_2$$

$$K_2 = -\frac{V_S}{R}$$

So

$$i(t) = \frac{V_S}{R}\left(1 + e^{-\frac{R}{L}t}\right)$$

And if we want $v_R(t)$:

$$v_R(t) = R \cdot i(t) = V_S\left(1 + e^{-\frac{R}{L}t}\right)$$

(similar to capacitor voltage in RC).

<u>Transient Circuit Analysis Step by Step Approach</u>

We still expect a solution

$$x(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$$

(x might be current or voltage)

Note as t -> ∞ $e^{\frac{-t}{\tau}} \to 0$ x(t) -> K_1

So if we find x(t) in steady-state (t -> ∞) with capacitor open circuit (for DC) and inductors short circuit (for DC), then x(t) = K_1.

Note at t = 0 x(t) = $K_1 + K_2$

Or at some time t_o:

$$x(t_o) = K_1 + K_2 e^{\frac{-t_o}{\tau}}$$

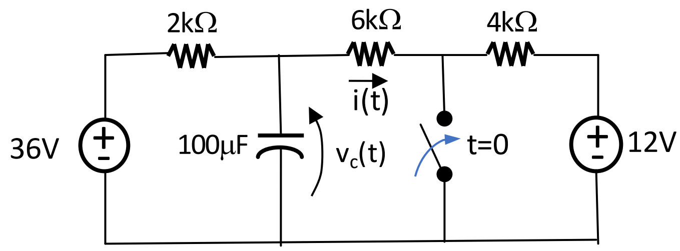
So if we know the initial value x(0) or the value at some time $x(t_o)$ we can also find K_2.

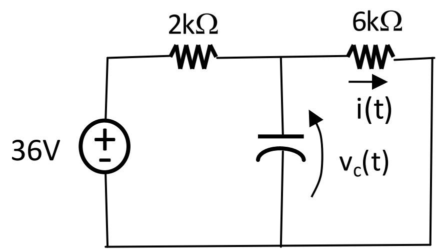
Often the 'initial condition' is just after a switch moves and is determined by the circuit before the switch changed.

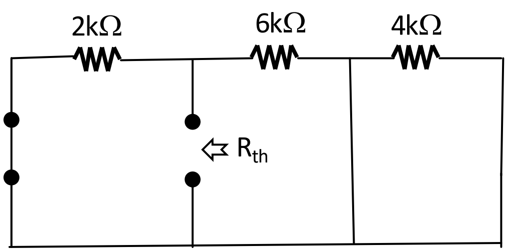
The time constant can be found from the Thevenin equivalent resistance at the terminals of C or L.

$\tau = R_{th}C$ or $\tau = \frac{L}{R_{th}}$

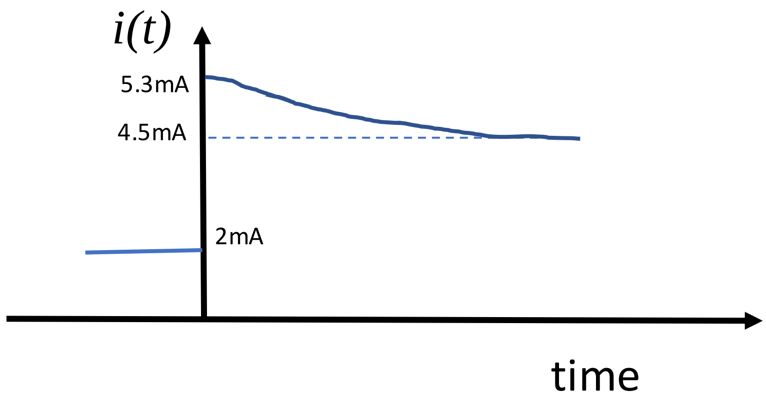
Example Find i(t) t > 0:



1) We expect: $v_c(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$
2) Initial capacitor voltage t = 0^- (steady state open circuit)
   - $i(0^-)$= (36V-12V)/(2k+6k+4k) = 2mA
   - $v_c(0^-)$ = 36V – 2mA X 2k = 32V
   - note i will change as soon as switch moves. v_c must be continuous.
3) At t=0^+ the switch connects 6k to ground. So new circuit is:
   - 

4) as t -> ∞ the capacitor is again open circuit
   - i(∞) = 36V/(6k+2k) = 4.5mA
   - $v_c(\infty)$= 4.5mA X 6k = 27V = K_1
5) From the initial condition we know that:
   - $v_c(0) = K_1 + K_2 e^0 = 32V \rightarrow K_2 = 32V - 27V = 5V$
6) To find τ we need the equivalent resistance seen by the capacitor
   - 

   - Note the 4k resistor is still shorted out by the switch.
   - Therefore R_{th} = 6k//2k = 1.5k
   - So $\tau = R_{th} \cdot C = 1.5k \cdot 100\mu F = 0.15s$
   - And our solution $v_c(t) = \left(27 + 5e^{\frac{-t}{0.15}}\right)V$
   - The current is then easily found by ohm's law as: $i(t) = \frac{v_c(t)}{6k} = 4.5 + 0.83e^{\frac{-t}{0.15}} mA$



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- Note the switch doesn't always change at t = 0 -> we may have an $e^{\frac{-t_o}{\tau}}$ term.
- e.g. $x(t) = x(\infty) + [x(t_o) - x(\infty)]e^{\frac{-(t-t_o)}{\tau}}$
- be very careful applying formula solutions!

Pulse Response

Many circuits are characterized by their response to a unit impulse or unit step function. We'll just consider a step for now

Unit step function $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

And a more general form: $u(t) = \begin{cases} 0 & t < t_o \\ 1 & t > t_o \end{cases}$

Note steps can be used to construct other functions, such as pulses:

$$v(t) = A[u(t) - u(t-T)]$$

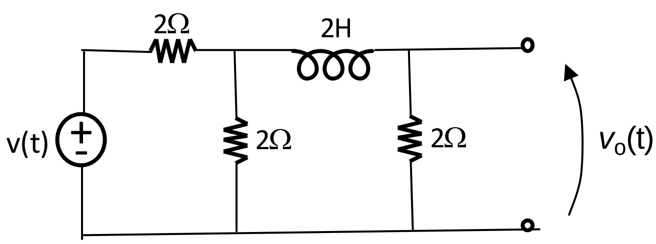
is a pulse from 0 to T

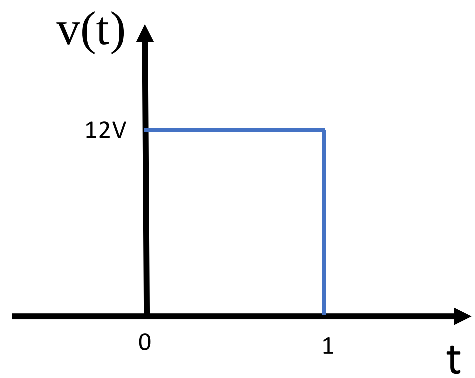
$$v(t) = A[u(t-t_o) - u(t-(t_o+T))]$$

is a pulse from t_o to t_o+T

Note we also write such functions piece wise.

Example what is $v_o(t)$ t > 0?

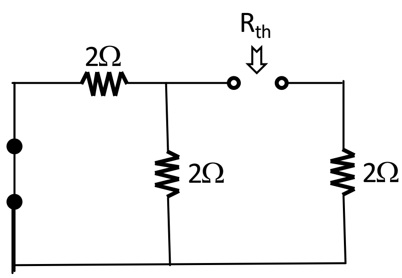




$v_o(t) = 0$ for $t < 0$ (no source)

at $t=0^+$ $i_o = 0$ (inductor) so $v_o = 0V$

as $t \rightarrow \infty$ if pulse did not end, inductor = short, $v_o = 1/3 \times 12V = 4V$



$R_{th} = 3\Omega$ **<u>****Note this is not obvious to a lot of students, but the 2ohm resistor on the right is in series with the other two in parallel. Remember a circuit is a circle!</u>**

so $\tau = {}^{L}/_{R} = {}^{2}/_{3} = 0.67s$

$$v_o(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$$

$$t \rightarrow \infty \quad K_1 = 4V$$

$$t \rightarrow 0 \quad v_o(t) = 0 = K_1 + K_2 \rightarrow K_2 = -K_1 = -4V$$

So

$$v_o(t) = 4\left(1 - e^{-\frac{3}{2}t}\right) \quad 0 < t < 1s$$

$$at\ t = 1s \quad v_o(1) = 4\left(1 - e^{-\frac{3}{2}}\right) = 3.11V$$

Now the pulse is turned off so eventually the voltage returns to zero. Therefore:

$$v_o(\infty) = 0 = K_1$$

The initial condition again gives us K_2 directly since K_1 is zero as 3.11V.

$$so \quad v_o(t) = 3.11e^{-\frac{3}{2}(t-1)} \quad for\ t > 1s$$

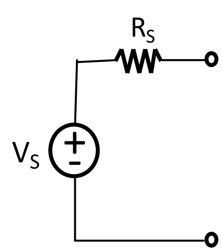
You may want to read RLC – we will come back to this later!

Design Example

One application of capacitors is to smooth-out sudden voltage changes, since voltage in a capacitor cannot change instantaneously.

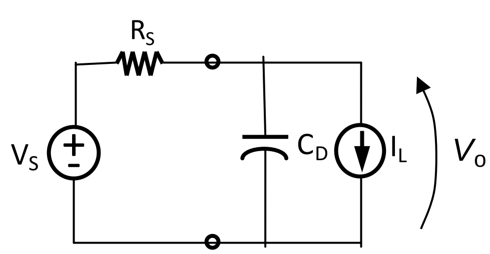
Capacitors are widely used for decoupling power supplies -> that is protecting circuits from rapid voltage fluctuations or disturbances.

Say we have a DC voltage source:

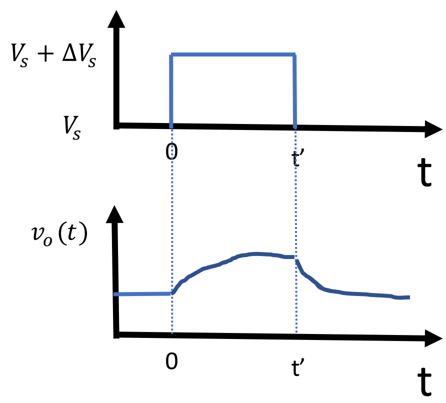


We would like a simple circuit to isolate a load from supply fluctuations.

We can do this with a single capacitor across the load:



Say our supply changes from $V_s \rightarrow V_s + \Delta V_s$ at time t = 0, and returns to V_s at t = t'



How will C_D influence the fluctuation in output voltage Δv_o?

Voltage on C_D: $v_o(t) = K_1 + K_2 e^{\frac{-t}{\tau}}$

$$at\ t = 0 \quad v_o(0) = K_1 + K_2 = V_S - I_L R_S$$

$at\ t = \infty \quad v_o(\infty) = K_1 = V_S + \Delta V_S - I_L R_S$ (for a step) used to get constants during a pulse

And the time constant is just $R_S C_D$. so:

$$v_o(t) = (V_S + \Delta V_S - I_L R_S) - \Delta V_S e^{\frac{-t}{R_S C_D}}$$

This applies until t', when the voltage supply returns to V_S.

$$v_o(t') = (V_S + \Delta V_S - I_L R_S) - \Delta V_S e^{\frac{-t'}{R_S C_D}}$$
$$= V_S - I_L R_S + \Delta V_o$$

Or

$$\Delta V_S - \Delta V_o = \Delta V_S e^{\frac{-t'}{R_S C_D}}$$
$$\frac{\Delta V_S}{\Delta V_S - \Delta V_o} = e^{\frac{t'}{R_S C_D}}$$
$$R_S C_D = \frac{t'}{\ln\left(\frac{\Delta V_S}{\Delta V_S - \Delta V_o}\right)}$$

Or

$$C_D = \frac{\frac{t'}{R_S}}{\ln\left(\frac{\Delta V_S}{\Delta V_S - \Delta V_o}\right)}$$

Note the choice of C_D depends on voltage change, not magnitude.

Given the expected size and length of voltage disturbances, source R and desired fluctuation in load voltage, we can choose C_D.

However good decoupling ($\frac{\Delta V_o}{\Delta V_S}$ $small$)requires large C_D value.