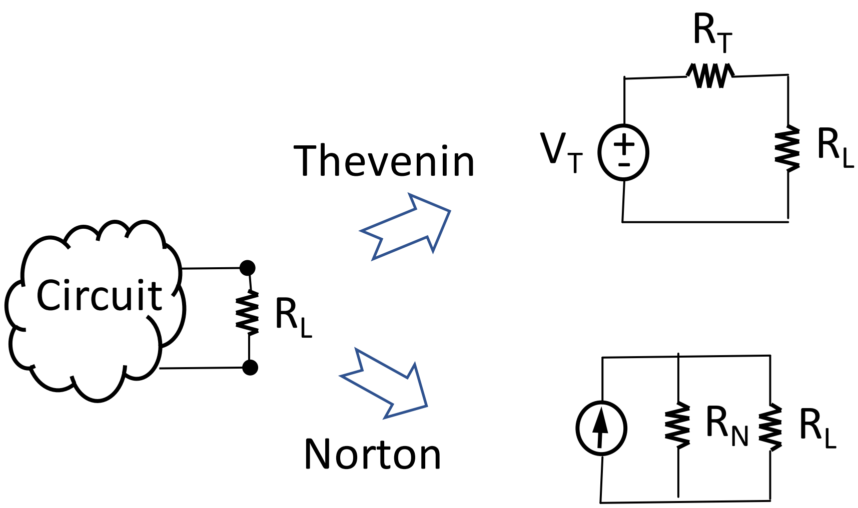
Thevenin's and Norton's Theorems

Often, we may use a relatively complex circuit to deliver power to a load.

Thevenin's theorem says that the entire circuit (exclusive of the load) can be replaced by an equivalent circuit containing only an independent voltage source in series with a resistor.

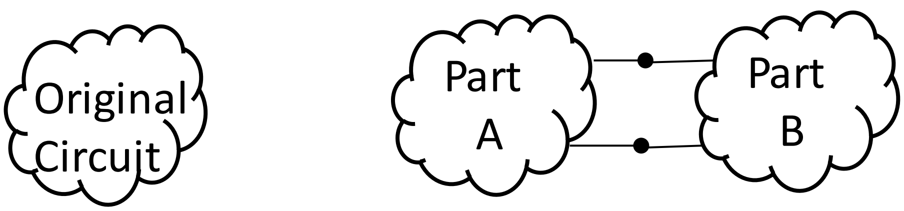
Norton's theorem says we can replace the driving circuit by an independent current source in parallel with a resistance.



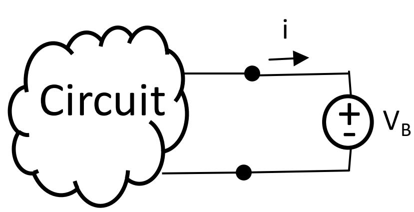
In both cases the current-voltage relation at the load stays the same. We can change the load value and the equivalent will accurately model the actual circuit!

Consider the Thevenin equivalent first.

Consider a complex circuit that can be split into two parts: A and B:



There is a current i between the circuits and a voltage V_B at the terminals. We could replace part B with a voltage source V_B without changing the voltage or current at the terminals.



If we were to apply superposition to the circuit, we could find the current i_B due to the new source V_B, and the current with V_B = 0 (short circuit) due to other sources in circuit A i_A, then the total will be

$$i = i_A + i_B$$

Setting all sources in A to zero we find:

$$i_B = -V_B/R_{TH}$$

where R_{TH} would be the resistance between the terminals of circuit A with all independent sources = 0, therefore:

$$i = i_A - V_B/R_{TH}$$

Now if we set $V_B = 0$ (we short the terminals) then we are left with:

$$i = i_{sc} = i_A$$

So

$$i = i_{SC} - V_B/R_{TH} \quad (***)$$

Now say we open circuit the terminals, then $V_B = V_{OC}$ and the current is equal to zero so:

$$i = 0 = i_{SC} - V_{OC}/R_{TH}$$

$$V_{OC} = R_{TH} \cdot i_{SC}$$

Thus, the ratio of the open circuit voltage and the short current is related by the resistance of circuit A with all sources zeroed out! We can therefore re-write (***) as:

$$i = V_{OC}/R_{TH} - V_B/R_{TH}$$

Solving for V_B:

$$V_B = V_{OC} - i \cdot R_{TH}$$

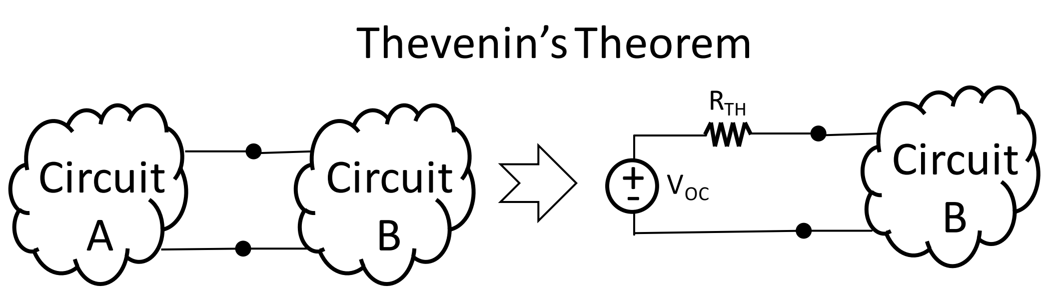
Look at the above equation. How would you draw it as a circuit? It is a voltage source in series with a resistor! The first term tells you the voltage when no current is drawn. The second term $i \cdot R_{TH}$ indicates that any current drawn from the circuit will reduce the output voltage -> real circuit.

<u>Thevenin</u>

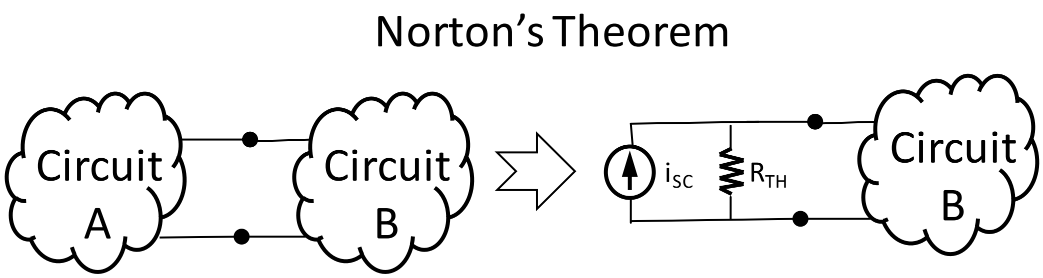
R_{TH} = resistance looking into the terminals of driving circuit A with all sources = 0 -> Theven Resistance.

V_{OC} = The voltage at the terminals of circuit A when the terminals are open circuit.

The circuit A can then be replaced by a source V_{OC} in series with a resistor R_{TH}:

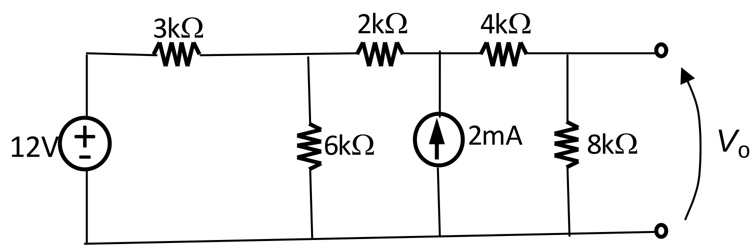


This indicates circuit A can also be replaced by a current source in parallel with a resistor R_{TH}.

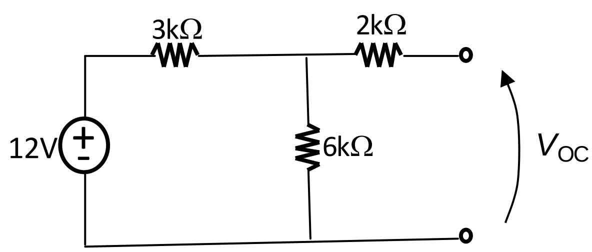


Note the two circuit equivalents are different representations of the same equations. Note they are equivalent only in terminal characteristics -> Norton dissipates power O.C.

Example: Use Thevenin's Theorem to find V_o



We can do this in stages. We will start by considering just the part to the left of the current source:

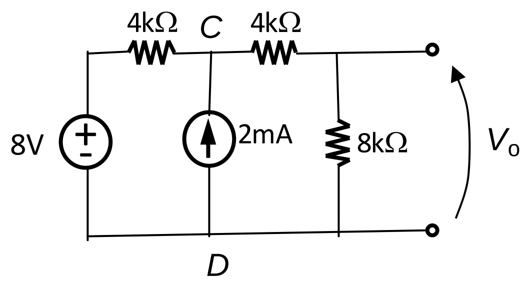


VTH = VOC = 12V X 6k/(6k+3k) = 8V (note this is a simple voltage divider between the 3k and 6k resistor… why?? We assume open circuit at the terminals so no current flow through the 2k resistor thus it doesn't appear in the equation).

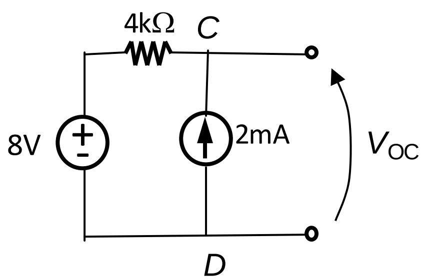
R_{TH} -> find the equivalent R between the terminals with sources = 0.

R_{TH} = 2k + 6k//3k = 4k (note 12V supply becomes a short circuit)

Our new circuit is:



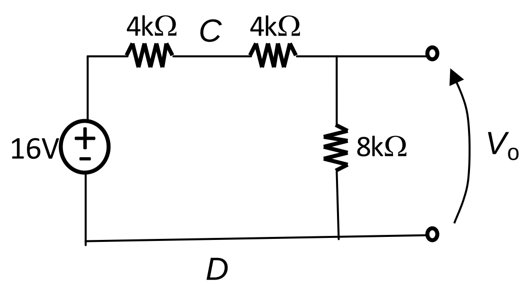
Now break the circuit at C/D:



The new V_{TH} = 8V + 2mA X 4k = 16V

New R_{TH} = 4k (current source open, voltage source short)

Our equivalent is now:

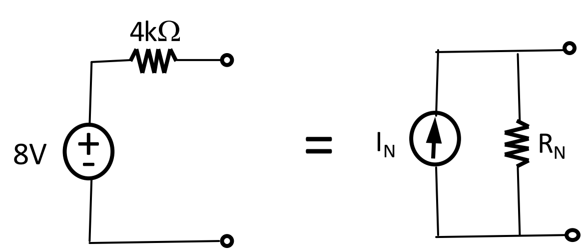


And Vo = 16V X 8k/(8k+4k+4k) = 8V and

R_{TH} = 8k//(4k+4k) = 4k (voltage source again a short circuit)

The equivalent could also be found using superposition of 2 sources.

Note:



I_N would be the short circuit current = 8V/4k = 2mA.

And R_N would provide the same O.C. voltage:

$8V = I_N \cdot R_N$ or $R_N = 8V/2mA = 4k$

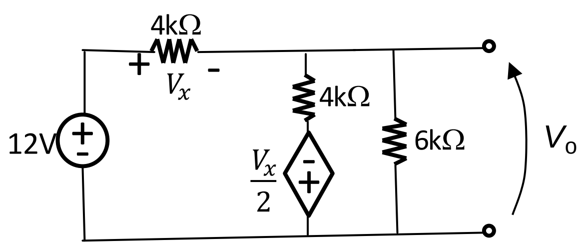
i.e. $R_N = R_{TH}$ -> source transformation.

Thevenin:

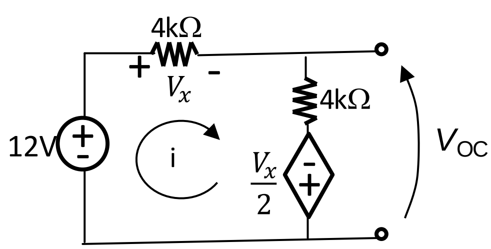
1) Remove the load, find the open circuit voltage.
2) Find the equivalent resistance between the output terminals with independent sources set to zero.
3) If there are dependent sources present R_{TH} may not be obvious so a 'test source' is connected to the output and R_{TH} is found from V_{TEST} / I_{TEST}. (ohm's law).
4) Re-attach the load and complete the analysis.

Thevenin Dependent and Independent Sources

Find V_o using Thevenin's Theorem:



We will once more do this in steps. First we will remove the 6k resistor:



Using KVL:

$12V - V_x - 4k \cdot i + V_x/2 = 0$

We also note that $V_x = 4k \cdot i$, therefore:

$12V - 4k \cdot i - 4k \cdot i + 2k \cdot i = 0$

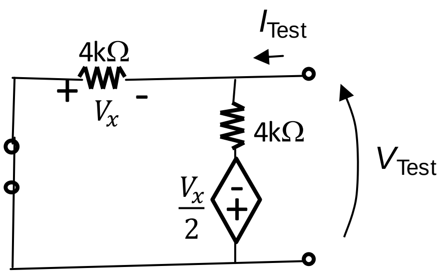
$i = 2mA$

We now note that:

$V_{OC} = 4k \cdot i - V_x/2$

$V_{OC} = 2k \cdot i = 2k \cdot 2mA = 4V$

We have found the open circuit voltage, next we need to find R_{TH} which isn't obvious because we have dependent sources! Thus, we will zero all independent sources and apply a test voltage and find the test current:



We note that $V_x = -V_{TEST}$

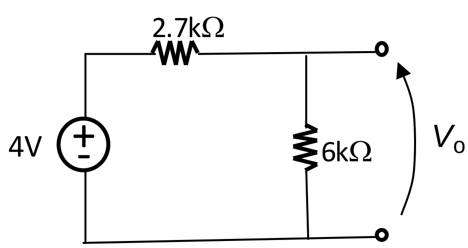
Therefore:

$I_{TEST} = V_{TEST}/4k + (V_{TEST}/2)/4K$

***Note a good test if you can keep track of negative signs! The 4k in series with dependent source sees a voltage of $+V_{TEST}/2$.

$V_{TEST}/I_{TEST} = (8k/3) = 2.7k$

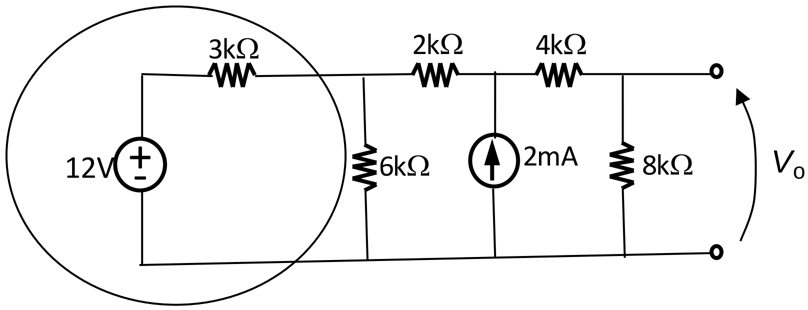
Now we can replace the 6k resistor with our simple model to give:



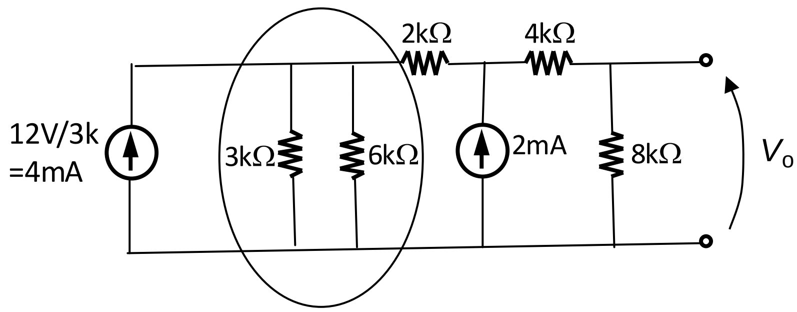
$V_o = 4V \times 6k/(2.7k+6k) = 2.75V$

<u>Source Transformation or Source Exchange</u>

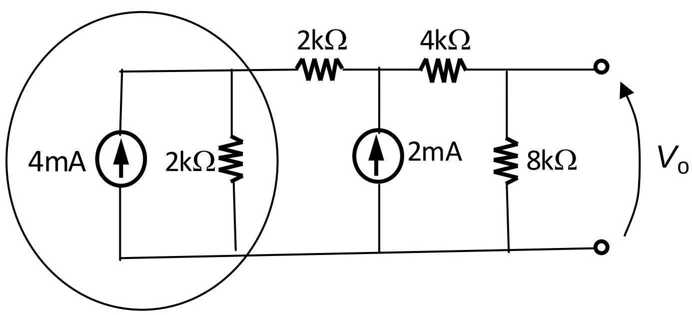
A current source in parallel with a resistor can be replaced by a voltage source in series with a resistor. This may simplify circuit analysis, for example allowing sources to be combined.



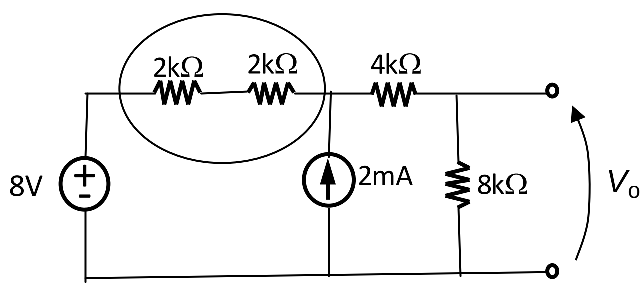
The circled elements can be replaced with their Norton equivalent:



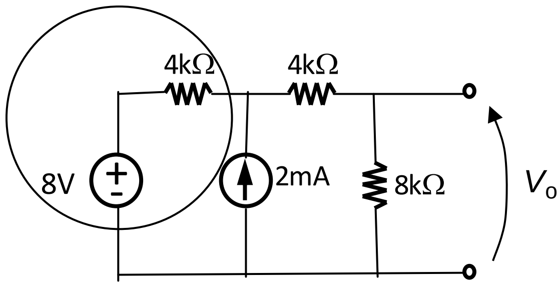
This puts two resistors directly in parallel for easy combining:



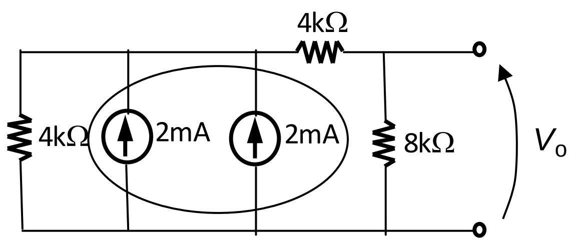
Now we can convert the two circled elements into a Thevenin equivalent:



Which means the two circled resistors can be easily combined:



Now converting back to Norton:



We can see that we can easily combine the two current sources now. Now using a current divider:

V_o = 4mA X (4k/16k) X 8k = 8V

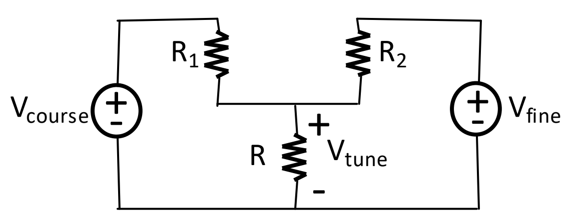
Of course we could also have kept reducing the circuit step by step as well…

Design Example

We would like to design a circuit to combine a course adjustment voltage and a fine adjustment voltage to give an overall tuning voltage:

V_{tune} = ½ V_{course} + 1/20 V_{fine}

Sum of two terms -> superposition of sources?



We want:

$$\frac{V_{tune\_c}}{V_{course}} = \frac{R//R_2}{R_1 + (R//R_2)} = \frac{1}{2}$$

$$R//R_2 = R_1$$

We also want:

$$\frac{V_{tune\_f}}{V_{course}} = \frac{R//R_1}{R_2 + (R//R_1)} = \frac{1}{20}$$

$$R//R_1 = \frac{1}{19}R_2$$

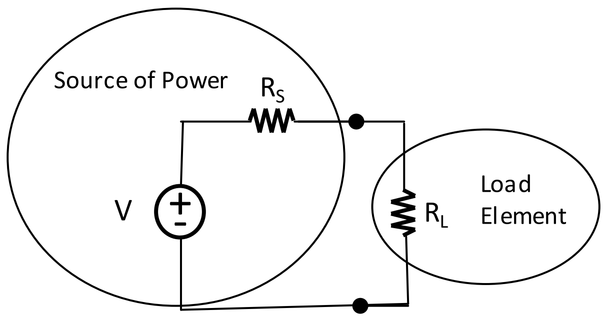
2 equations -> 3 unknowns

Another constraint? Power dissipation?

Otherwise choose ‘reasonable values’ i.e. pick at least one resistor arbitrarily.

Maximum Power Transfer

In some applications we are interested in transferring the maximum possible power to a load element or circuit. The Thevenin equivalent circuit can aid us with this.



$$P_{load} = i^2 R_L = \left(\frac{V}{R_S + R_L}\right)^2 R_L$$

What value of R_L makes this a maximum?

$$\frac{dP_{load}}{dR_L} = 0 = \left(\frac{V}{R_S + R_L}\right)^2 - 2R_L \frac{V^2}{(R_S + R_L)^3}$$

$$\frac{2R_L}{R_S + R_L} = 1$$

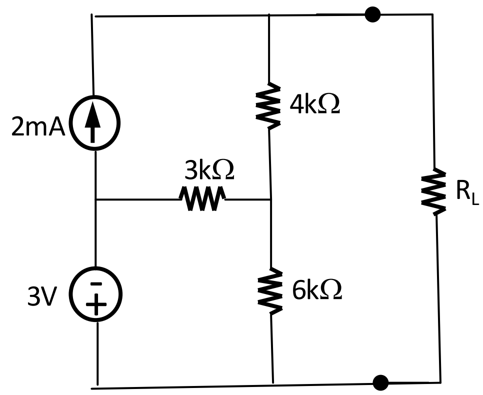
$$2R_L = R_S + R_L$$

$$R_L = R_S$$

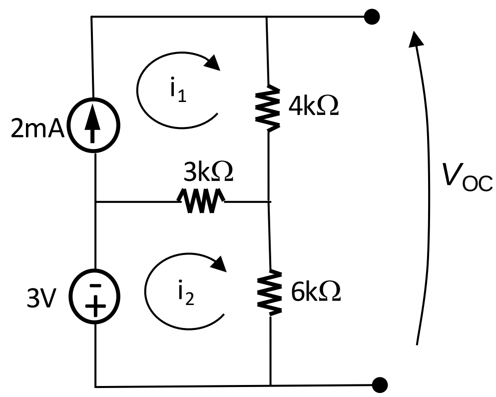
Maximum power transfer occurs when the load resistance matches the source resistance.

Example

Find R_L for maximum power transfer.



We start by removing R_L and finding the Thevenin equivalent. Note we only need R_{TH} to find R_L and we don't need V_{OC} unless we want to know the VALUE of the power transferred to the load.



$i_1 = 2mA$

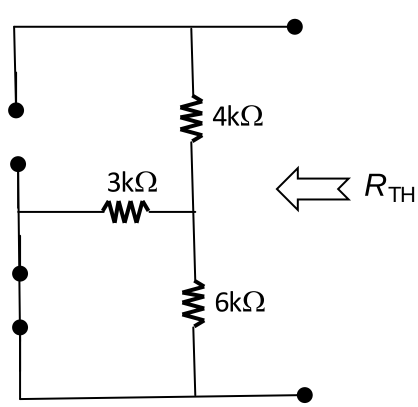
$-3V - 3k\ (i_2 - i_1\ ) - 6ki_2 = 0$

$3V\ -6V + 9k\ i_2 = 0$

$i_2 = 3V/9k = 0.33mA$

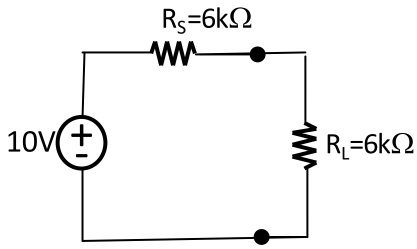
Then $V_{OC} = 6ki_2 + 4ki_1 = 6k \text{ X } 0.33mA + 4k \text{ X } 2mA = 10V$

Now to find R_{TH}, voltage source -> 0 (short), current source -> open



R_{TH} = 4k + 6k//3k = 6k

So R_L = 6k for max power.



P = i^2 6k = $(10V/12k)^2$ 6k = 4.17mW

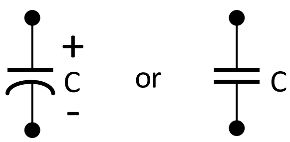
Capacitors

We now have all the basic tools for circuit analysis, but so far the only components we have are sources and elements that dissipate energy -> resistors.

We can do some interesting things with elements that store energy -> inductors and capacitors

Capacitors - A capacitor is a circuit element composed of two conducting surfaces separated by a non-conducting dielectric material.

Schematic symbol:



Often capacitor electrodes are rolled to give a high capacitance to volume ratio. Different dielectrics are used depending on the application.

-ceramic

- electrolytic (Al, Ta)

-mica

-polyester

And some new high capacity materials.

Capacitance is charge / potential

Coulombs/volt = farads (F)

(after Michael Faraday)

Typical values are μF to pF, although recently energy storage capacitors up to a few 100F have become available.

- ➔ Voltage rating -> breakdown or catastrophic failure.

Parallel plate capacitor is commonly analyzed

$$C = \frac{\epsilon \cdot A}{d}$$

Two plates of area A are separated by distance d by a dielectric with permittivity ϵ.

Consider the 100F double layer capacitor mentioned above with an air gap $\epsilon = \epsilon_o$ of distance 10^{-4}m.

$$100F = \frac{8.85X10^{-12} \cdot A}{10^{-4}m}$$

A = 1.148X10^9m^2 or 443 square miles!

Note capacitance relates applied voltage to stored charge:

$q = CV$ -> energy stored in electric field

And we know current

$$i = \frac{dq}{dt} = \frac{d(Cv)}{dt}$$

So

$$i = C\frac{dv}{dt}$$

For constant capacitance or turning it around:

$$dv = \frac{1}{C} idt$$

And

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

And since we may only know a condition at some initial time

$$v(t) = \frac{1}{C}\int_{-\infty}^{t_o} i(t)\,dt + \frac{1}{C}\int_{t_o}^{t} i(t)\,dt$$

$$v(t) = v(t_o) + \frac{1}{C}\int_{t_o}^{t} i(t)\,dt$$

Power delivered to the capacitor

$$p(t) = v(t) \cdot i(t)$$

$$= v(t)C\frac{dv(t)}{dt}$$

And the energy can be found by integrating:

$$p(t)dt = Cv(t)dv$$

$$\omega_c(t) = \int_{-\infty}^{t} p(t)\,dt$$

$$= C\int_{-\infty}^{t} v(t)\,dv(t)$$

$$\omega_c(t) = \frac{1}{2}Cv^2(x)\Bigg|_{v(-\infty)}^{v(t)} = \frac{1}{2}Cv^2(t)$$

Also from q = CV :

$$\omega_c(t) = \frac{1}{2}\frac{q^2(t)}{C}$$

Since $i = C\frac{dv}{dt}$, for a constant (DC) voltage there is no capacitor current in the steady state -> a capacitor is an open circuit to DC or 'blocks' DC.

****think about this for a moment…what is a capacitor physically? -> it is a broken wire!!!

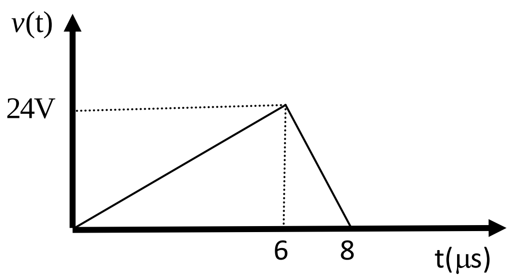
Note however there is an initial **transient** current when DC is switched on or off. Also since

$$p(t) = v(t)C\frac{dv(t)}{dt}$$

An instantaneous voltage change would require infinite current and power -> not physically possible. -> capacitor voltage must be continuous.

Example:

The voltage across a 5μF capacitor is given by:



What is the current?

$$i(t) = C\frac{dv(t)}{dt}$$

$$i(t) = 5\mu F\frac{24V}{6\mu s} = 20\mu A$$

For 0 ≤ t ≤ 6μs.

$$i(t) = 5\mu F\frac{-24V}{2\mu s} = -60\mu A$$

For 6μs ≤ t ≤ 8μs.

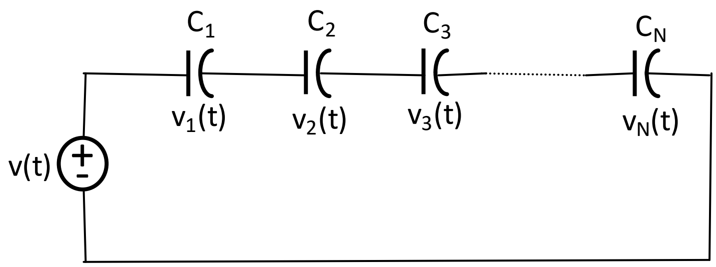
i(t) = 0μA for t > 8 μs.

Energy stored at t = 6μs depends only on voltage at that instant (not how it got there).

$$E = \frac{1}{2}Cv^2 = \frac{1}{2}\cdot 5\mu F \cdot 24V^2 = 1.44mJ$$

Series Capacitors

We can find how to combine elements from KVL, KCL, and the I-V relationship.



KVL: $v(t) = v_1(t) + v_2(t) + v_3(t) + \ldots + v_N(t)$

And

$$v_i(t) = v_i(t_o) + \frac{1}{C_i}\int_{t_o}^{t} i(t)\,dt$$

(same current in each capacitor) So:

$$v(t) = \sum_{i=1}^{N} v_i(t_o) + \sum_{i=1}^{N} \frac{1}{C_i}\int_{t_o}^{t} i(t)\,dt$$

$$= \frac{1}{C_S}\int_{t_o}^{t} i(t)\,dt + v(t_o)$$

Where

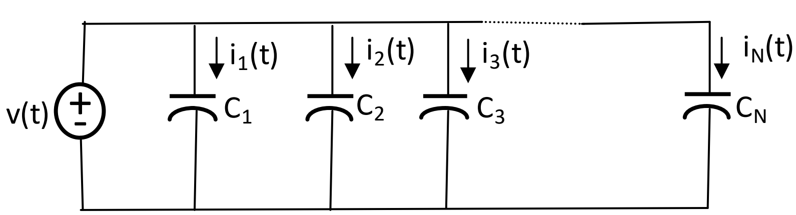
$$v(t_o) = \sum_{i=1}^{N} v_i(t_o)$$

And

$$\frac{1}{C_S} = \sum_{i=1}^{N} \frac{1}{C_i} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$$

- ➔ Same current, so each capacitor acquires the same charge in a given time.

Parallel Capacitors



KCL: $i(t) = i_1(t) + i_2(t) + i_3(t) + \ldots + i_N(t)$

And

$$i_i(t) = C_i \frac{dv_i(t)}{dt}$$

But $v_i(t) = v(t)$ for all C_i so:

$$i(t) = C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + C_3 \frac{dv(t)}{dt} + \cdots + C_N \frac{dv(t)}{dt}$$

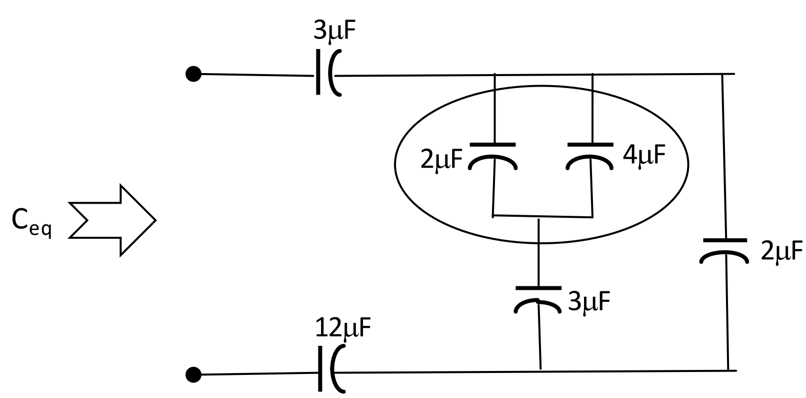
So

$$i(t) = C_P \frac{dv(t)}{dt}$$

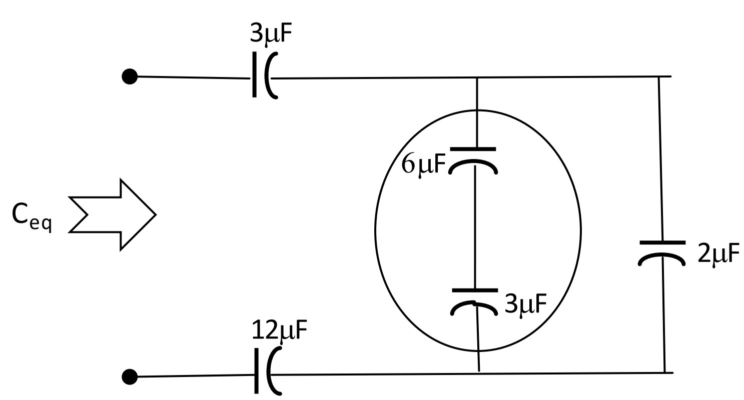
Where:

$$C_P = \sum_{i=1}^{N} C_i$$

<u>Example Find the Equivalent Capacitance</u>

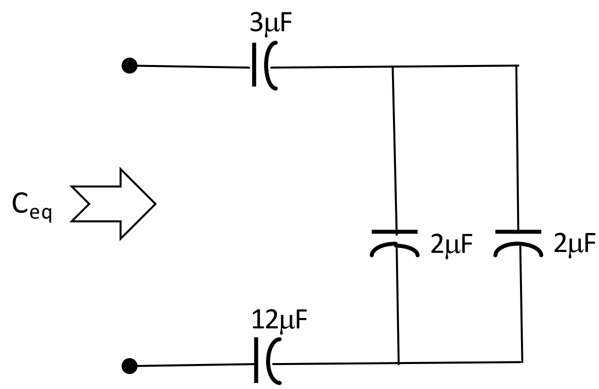


We first note that the two circled caps are in parallel $C_p = 2\mu F + 4\mu F = 6\mu F$.



We next note that these two circled caps are in series: $C_s = (1/6\mu F + 1/3\mu F)^{-1} = 2\mu F$

This leaves us with:



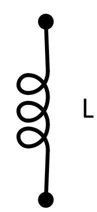
We note the two parallel caps have an overall capacitance of 2μF + 2μF = 4μF.

Thus we are left with a 3 μF, 4μF, and 12μF in series:

$C_{eq} = (1/12 + 1/4 + 1/3)^{-1} = 1.5\mu F$

Inductors

An inductor consists of a conducting wire usually in the form of a coil.



Typically categorized by the core material around which the wire is wound.

- ➔ Air and nonmagnetic materials
- ➔ Iron
- ➔ Ferrite (iron, iron oxide -> ferromagnetic)

As with capacitors materials depend on value and loss requirements.

Inductance is measured in Henrys (after Joseph Henry) -> volt/s per ampere.

$$v(t) = L\frac{di(t)}{dt}$$

Note that current cannot change instantaneously. We can also write this formula as:

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt = i(t_o) + \frac{1}{L}\int_{t_o}^{t} v(t)\,dt$$

Typical values are μH to mH.

Inductors store energy in a magnetic field created by the current. Power is still given by:

$$p(t) = v(t) \cdot i(t)$$
$$= L\frac{di(t)}{dt} \cdot i(t)$$

Or energy is given by:

$$\omega_L(t) = \int_{-\infty}^{t} p(t)\,dt$$
$$\omega_L(t) = \int_{-\infty}^{t} L\frac{di(t)}{dt} \cdot i(t)\,dt = \frac{1}{2}Li^2(t)$$

Short circuit for DC (ideal, some small R). THINK about this, and inductor is just a wire!!!

Note:

$$p(t) = L\frac{di(t)}{dt} \cdot i(t)$$

So power is proportional to the instantaneous change in i(t). If di/dt goes to infinity power goes to infinity which is not possible. Therefore, an inductor requires continuity of current.

Inductors are indispensable in tuned circuits and many power applications. – motors, transformers and solenoids (actuators).

Example

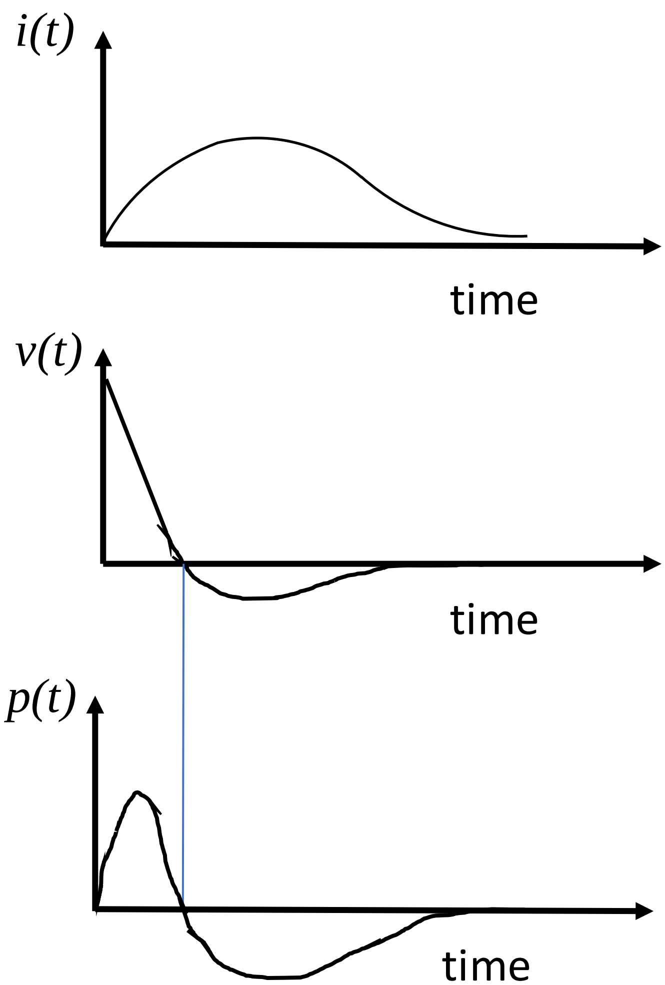
The voltage measured across a 200mH is $v(t) = (1\text{-}3t)e^{-3t}$ mV for $t \geq 0$ and zero for $t < 0$. What is the current and power?

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt$$
$$= \frac{10^3}{200}\int_{0}^{t} (1-3t)e^{-3t}\,dt$$
$$= 5\left(\int_0^t e^{-3t}\,dt - 3\int_0^t te^{-3t}\,dt\right)$$
$$= 5\left[-\frac{1}{3}e^{-3t}\Big|_0^t - 3\left(\left(-\frac{1}{3}te^{-3t}\right) - \left(\frac{1}{9}e^{-3t}\right)\right)\Big|_0^t\right]$$
$$= 5\left[-\frac{1}{3}e^{-3t} + \frac{1}{3} + (te^{-3t} - 0) + \frac{1}{3}e^{-3t} - \frac{1}{3}\right]$$
$$i(t) = 5te^{-3t}mA \quad t \geq 0$$
$$p(t) = v(t) \cdot i(t) = 5te^{-3t} \cdot (1-3t)e^{-3t} = 5t(1-3t)e^{-6t}\mu W$$

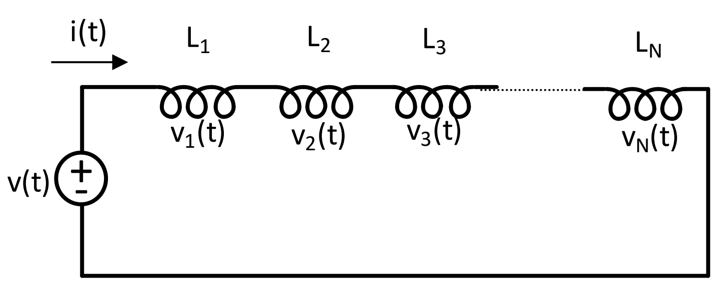
Notice i(t) is positive for t > 0.

v(t) is positive for di/dt > 0 and negative for di/dt < 0.

i(t) positive, p(t) has the sign of v(t) starting positive (storing) and ending negative (releasing).



Series Inductors



KVL:

$$v(t) = v_1(t) + v_2(t) + v_3(t) + \cdots v_N(t)$$

$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \cdots + L_N \frac{di}{dt}$$

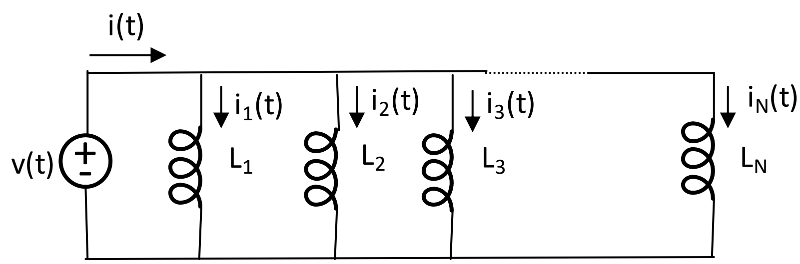
Since i is the same for each:

$$v(t) = \sum_{i=1}^{N} L_i \cdot \frac{di}{dt} = L_S \cdot \frac{di}{dt}$$

Where:

$$L_S = \sum_{i=1}^{N} L_i$$

Parallel Inductors



KCL: $i(t) = i_1(t) + i_2(t) + i_3(t) + \ldots + i_N(t)$

where

$$i_i(t) = i_i(t_o) + \frac{1}{L_i} \int_{t_o}^{t} v(t)\, dt$$

v(t) is the same across each L_i so:

$$i(t) = \sum_{i=1}^{N} i_i(t_o) + \sum_{i=1}^{N} \frac{1}{L_i} \int_{t_o}^{t} v(t)\, dt$$

$$= \frac{1}{L_P} \int_{t_o}^{t} v(t)\, dt + i(t_o)$$

Where

$$i(t_o) = \sum_{i=1}^{N} i_i(t_o)$$

And

$$\frac{1}{L_P} = \sum_{i=1}^{N} \frac{1}{L_i} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}$$

Similar to resistors.