Circuits with Dependent Sources

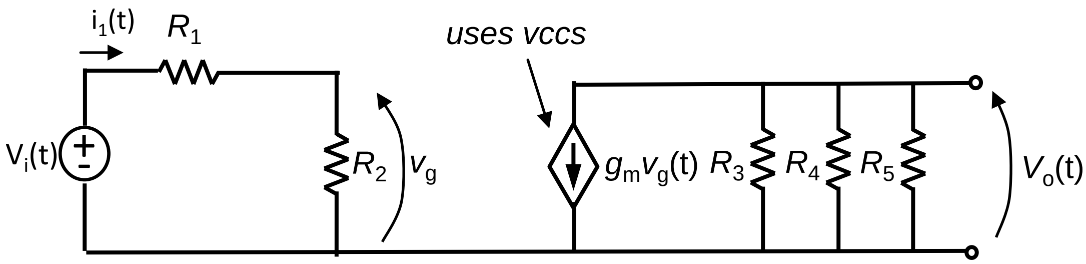
Dependent sources will become extremely important for modeling bipolar transistors and field effect transistors. Transistors are in turn building blocks of analog and digital circuits.

If we include dependent sources our analysis techniques don't really change.

Follow single loop KVL or single node pair KCL strategies

Example

An equivalent circuit (small signal AC) for a FET common-source amplifier or Bipolar (BJT) common-emitter amplifier is shown below:



The amplifier 'gain' is the ratio of $v_o(t)/v_i(t)$

This looks odd as there are two circuits, but it is fairly simple to analyze.

Input circuit is a single loop or voltage divider

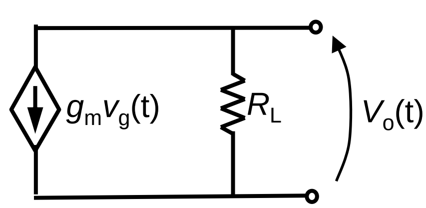
$$v_i(t) = i_i(t)(R_1 + R_2)$$
$$v_g(t) = i_i(t)R_2$$

So:

$$v_g(t) = v_i(t)\frac{R_2}{R_1 + R_2}$$

The output circuit is a single node-pair, and the resistors can be combined:

$$\frac{1}{R_L} = \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}$$



$$v_o(t) = -g_m v_g(t) R_L$$

We can incorporate the expression for $v_g(t)$:

$$v_o(t) = -g_m R_L \frac{R_2}{R_1 + R_2} v_i(t)$$

So the voltage gain:

$$\frac{v_o(t)}{v_i(t)} = \frac{-g_m R_L R_2}{R_1 + R_2}$$

Typical values: $R_1 = 100\Omega$, $R_2 = 1k\ \Omega$, $R_3 = 50k\ \Omega$, $R_4 = R_5 = 10k\ \Omega$, $g_m = 0.04s$ which gives $v_o(t)/v_i(t) = -175.3$

Note that the output is negative (inverted) which can also be meant to indicate an ac phase shift of 180 degrees.

Irwin 2.9 gives a nice description of modern resistor technologies

Surface mount (no leads) -> thick film

➔ Thin film

Integrated -> diffused

Nodal and Loop Analysis

We would like to now extend our capabilities to include multi-node and multi-loop circuits.

We will use KCL for node analysis

KVL for loop or mesh analysis

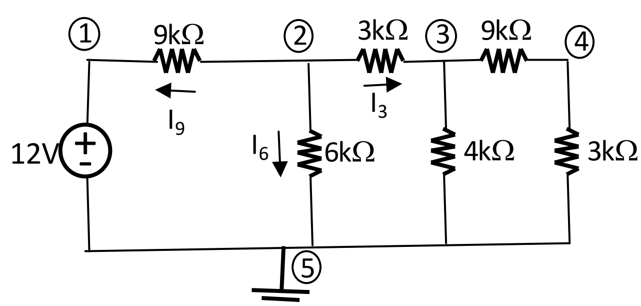
These techniques can be applied to very complex circuits but will result in a system of equations that must be solved to determine the variables. Well suited to computer aided analysis

Nodal Analysis

Variables are chosen to be node voltages, defined with respect to a common point in the circuit.

The reference is often chosen as the node with the most branches connected and is commonly called ground.

Choose node voltages positive relative to the reference.



e.g. label nodes

choose a reference

assign voltages V_1, V_2, V_3, V_4 (and $V_5 = 0$, reference)

write KCL for each node, with currents in terms of node voltages

for node 2 (summing currents flowing OUT of node 2):

$$\left(\frac{V_2 - V_1}{9k\Omega}\right) + \frac{V_2}{6k\Omega} + \left(\frac{V_2 - V_3}{3k\Omega}\right) = 0$$

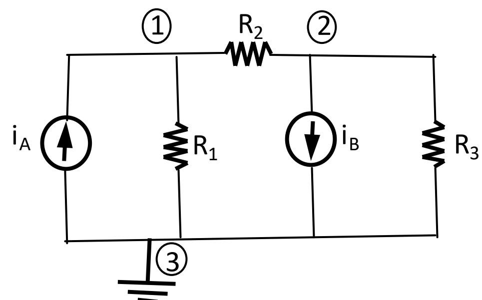
Do this for each node

Solve the system for node voltages

- ➔ Voltage sources fix node voltage differences
- ➔ Current sources fix branch currents

NOTE: You COULD solve this circuit without the need for any simultaneous equations. If you look carefully there are three nested voltage dividers so a smaller faction of the 12V appears at node 2, then a smaller faction appears at node 3 and a final lowest voltage appears at node 4.

Circuits with Only Independent Current Sources



Number nodes, select a reference, assign voltages V_1, V_2. Apply KCL to node 1:

$$i_A - \frac{V_1}{R_1} - \frac{V_1 - V_2}{R_2} = 0$$

(text uses conductance, usually easier just to always use resistance)

$(V_1-V_2)/R_2$ is a current from 1 -> 2 if everything is positive. $V_3 = 0$ so we don't bother writing it.

This can be arranged:

$$i_A = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_1 - \frac{1}{R_2}V_2$$

Now consider node 2:

$$\frac{V_1 - V_2}{R_2} - i_B - \frac{V_2}{R_3} = 0$$

Or

$$i_B = \frac{V_1}{R_2} - \left(\frac{1}{R_2} + \frac{1}{R_3}\right) V_2$$

So we have 2 equations for 2 unknown node voltages.

Say we have values, i_A = 1mA, i_B = 4mA, R_1 = 12kΩ, R_2 = 6kΩ, R_3 = 6kΩ

Our equations are:

$$1mA = \left(\frac{1}{12k} + \frac{1}{6k}\right) V_1 - \frac{1}{6k} V_2$$

$$4mA = \frac{V_1}{6k} - \left(\frac{1}{6k} + \frac{1}{6k}\right) V_2$$

Or

$$\frac{V_1}{4k} - \frac{V_2}{6k} = 1mA$$

$$\frac{V_1}{6k} - \frac{V_2}{3k} = 4mA$$

Replace V_1 in the second equation:

$$\frac{2}{3} V_2 + 4 - 2V_2 = 24$$

$$-\frac{4}{3} V_2 = 20$$

Or V_2 = -15V and

$$V_1 = \frac{2}{3}(-15) + 4 = -6V$$

This is easy for 2 unknowns but becomes complicated for more.

Once you know these voltages, you know all the currents.

Note these equations have the form G ● V = I so V = G^{-1} ● I which can be solved using matrix methods, and

$$G^{-1} = \frac{1}{|G|} adj(G)$$

Where

$$adj \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

And

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Or you can use Cramer's rule:

For Ax = b then $x_i = |A_i|/|A|$ where A_i is matrix with i_{th} column replaced by b.

In this case:

$$G=\begin{bmatrix} \frac{1}{4k} & \frac{-1}{6k} \\ \frac{1}{6k} & \frac{-1}{3k} \end{bmatrix}$$

$$|G|=\frac{1}{4k}\left(\frac{-1}{3k}\right)-\frac{1}{6k}\left(\frac{-1}{6k}\right)$$

$$=\frac{-1}{12k^2}+\frac{1}{36k^2}=\frac{-2}{36k^2}=\frac{-1}{18k^2}$$

$$|G_2|=\begin{bmatrix} \frac{1}{4k} & 1mA \\ \frac{1}{6k} & 4mA \end{bmatrix}=\frac{4mA}{4k}-\frac{1mA}{6k}=\frac{5mA}{6k}$$

So

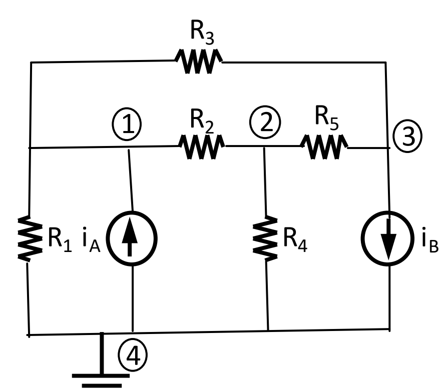
$$V_2=-\left(\frac{\frac{5mA}{6k}}{\frac{1}{18k^2}}\right)=-15V$$

$$|G_1|=\begin{bmatrix} 1mA & \frac{-1}{6k} \\ 4mA & \frac{-1}{3k} \end{bmatrix}=\frac{-1mA}{3k}+\frac{4mA}{6k}=\frac{1mA}{3k}=6V$$

Of course this might also be solved using a computer (Matlab, Maple, Mathematica…) or even some scientific calculators.

If you do this, check your result (using for example KVL) when you are done!

There is some symmetry in the matrix form and with some practice you might be able to write the matrix 'by inspection'. I encourage you to write node equations!



$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & \frac{-1}{R_2} & \frac{-1}{R_3} \\ \frac{-1}{R_2} & \frac{1}{R_2}+\frac{1}{R_4}+\frac{1}{R_5} & \frac{-1}{R_5} \\ \frac{-1}{R_3} & \frac{-1}{R_5} & \frac{1}{R_5}+\frac{1}{R_3} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} i_A \\ 0 \\ i_B \end{bmatrix}$$

Dependent current sources may destroy symmetry

Independent voltage sources usually fix a node voltage and make analysis simpler

It can be difficult to include a voltage source in a node equation - you don't know the current .

There are lots of examples in the text and we will tackle a few.

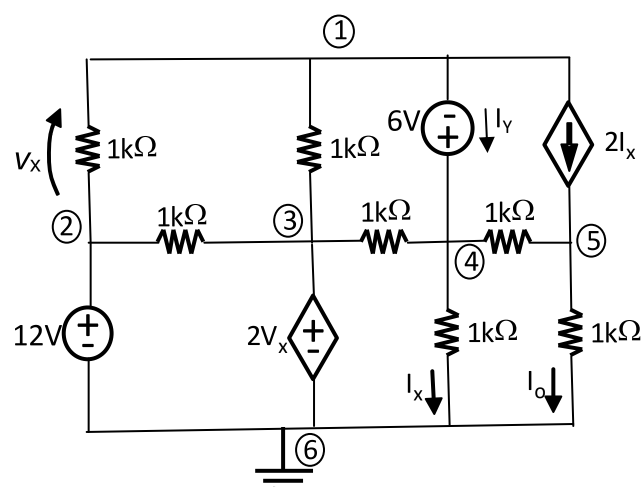
What about Gaussian Elimination?

- ➔ Put matrix in triangular form by eliminating one variable from each row by adding/subtracting rows.
- ➔ Once in triangular form, back-substitute.

For small systems the method doesn't matter much. For large systems Gaussian elimination is generally considered more efficient than Cramer's rule.

<u>Example</u>

Find I_o:



We first number nodes so they are easy to keep track of and decide on a reference (node 6 in this case). We also note that:

1) $V_6 = 0$ (reference)
2) $V_2 = 12V$
3) $V_x = V_1 - 12V$ (this voltage is needed as it controls a voltage source).
4) $V_4 = V_1 + 6V$ (note these two nodes are connected by a voltage source so we are only going to need to find one and we immediately know the other).

5) $V_3 = 2V_x = 2V_1 - 24V$ (voltage source connected from ground).
6) $I_x = V_4/1k = (V_1 + 6)/1k$ (this current is needed as it controls a current source).

Ok once we feel we have written down everything we can by inspection we can proceed with more rigorous KCL for all the nodes:

Node 1:

$(V_1 - V_2)/1k + (V_1 - V_3)/1k + 2I_x + I_Y = 0$

$(V_1 - 12)/1k + (V_1 - V_3)/1k + (2V_1 + 12)/1k + I_Y = 0$

Simplifying:

$-I_Y = 4V_1/1k - V_3/1k$

$I_Y = -(2V_1 - 24)/1k$ Equation (1)

Node 4:

$I_Y - V_4/1k - (V_4 - V_3)/1k - (V_4 - V_5)/1k = 0$

$I_Y - (V_1 + 6)/1k - ((V_1 + 6) - V_3)/1k - ((V_1 + 6) - V_5)/1k = 0$

$I_Y = (3V_1 + 18)/1k - V_3/1k - V_5/1k$

$I_Y = (3V_1 + 18)/1k - (2V_1 - 24V)/1k - V_5/1k$

$I_Y = (V_1 + 42)/1k - V_5/1k$ Equation (2)

Node 5:

$2I_x - (V_5 - V_4)/1k - V_5/1k = 0$

$2I_x + V_4/1k - 2V_5/1k = 0$

$(2V_1 + 12)/1k + V_4/1k - 2V_5/1k = 0$

$2V_1 + V_4 - 2V_5 = -12$

$3V_1 - 2V_5 = -18$ Equation (3)

Ok so we have 3 equations and we don't know V_1, I_Y, and V_5 so we can solve:

First we will remove I_Y by noting (1) and (2) are equal:

$-(2V_1 - 24)/1k = (V_1 + 42)/1k - V_5/1k$

$-2V_1 - 24 = V_1 + 42 - V_5$

$-3V_1 + V_5 = 66$

$V_5 = 3V_1 + 66$

Now sub this back into (3):

$3V_1 - 2(3V_1 + 66) = -18$

$-3V_1 = 114$

$V_1 = -38V$

Now $V_5 = -48V$

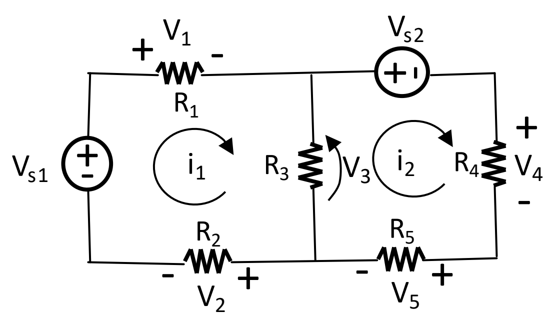
Therefore $I_o = -48V/1k = -48mA$

Loop Analysis

In this technique we will use loop currents as variables and write all branch voltages in terms of loop currents. We can then write KVL around each loop to generate a set of equations.

Solution will provide all loop currents in the circuit.

Voltage sources could cause confusion in nodal analysis (undefined current). Similarly current sources can be a challenge for loop analysis (undefined voltage).



Define loop currents - good idea to be consistent e.g. clockwise.

Write KVL for each loop

Loop 1:

$$V_{s1} - i_1 R_1 - (i_1 - i_2)R_3 - i_1 R_2 -= 0$$

Loop 2:

$$-V_{s2} - i_2 R_4 - i_2 R_5 + (i_1 - i_2)R_3 = 0$$

We now have two equations and two unknown currents so we could solve for i_1 and i_2.

Note resistors between 2 loops carry both currents and the net current is given by the sum. You may fix voltage polarity $V_3 = (i_1 - i_2)R_3$.

Loop1 -> $+V_3$

Loop2 -> $-V_3 = (i_2 - i_1)R_3$

Collecting terms:

$$i_1(R_1 + R_2 + R_3) - i_2R_3 = V_{s1}$$

$$-i_1R_3 + i_2(R_1 + R_2 + R_3) = -V_{s2}$$

Or:

$$\begin{bmatrix} R_1 + R_2 + R_3 & -R_3 \\ -R_3 & R_1 + R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_{s1} \\ -V_{s2} \end{bmatrix}$$

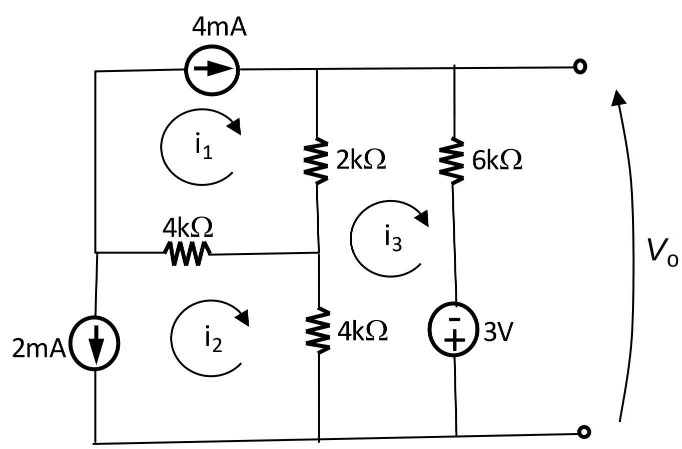
Which looks similar to our nodal analysis equations.

Due to the symmetry you may be able to write equations by inspection, but I don't recommend it as it can be impossible to track errors.

Independent Current Sources - voltage drop is unknown, but analysis can be simplified as a loop current may be fixed by the source.

Example:

Find V_o:



Define loop currents --- we write KVL or a constraint $i_1 = 4mA$, $i_2 = -2mA$

This leaves only one KVL equation:

$$4k(i_2 - i_3) + 2k(i_1 - i_3) - 6ki_3 + 3V = 0$$

$$12ki_3 - 4k(-2mA) - 2k(4mA) = 3V$$

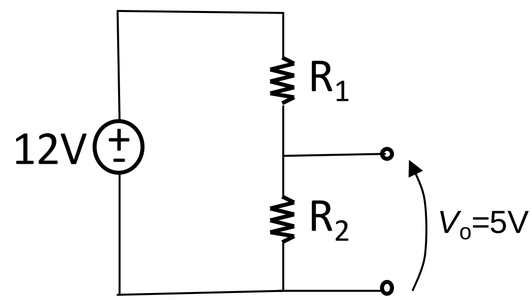
$$12ki_3 = 3V + 8V - 8V$$

$$i_3 = \frac{3V}{12k} = 0.25mA$$

Therefore, $V_o = -3V + 0.25mA \cdot 6k = -3V + 1.5V = -1.5V$

<u>Challenge:</u>

a) Design a circuit to provide a 5V open circuit output from a 12V battery. Power consumed (absorbed) in the circuit should be < 1W.



We first note that the power consumed by the circuit is 1W therefore:

$$P = \frac{V^2}{R_1 + R_2}$$

$$R_1 + R_2 = \frac{V^2}{P} = \frac{12^2}{1W} = 144\Omega$$

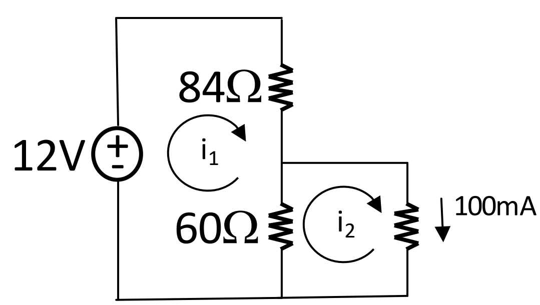
Next we use voltage divider to find the output voltage (can't remember voltage divider? No worries, what current flows through the two resistors? Know the current, then you can find the voltage across R_2):

$$V_o = 12V \frac{R_2}{R_1 + R_2} = 5V$$

$$R_2 = \frac{5V}{12V}(R_1 + R_2) = \frac{5}{12}(144) = 60\Omega$$

Therefore, R_1 = 144-60=84Ω.

b) A load drawing 100mA is connected to your 5V output terminals. What is the output voltage now?



We note right away that i_2 = 0.1A. Therefore, we only need to write KVL for loop1:

$$12V - 84i_1 - 60(i_1 - i_2) = 0$$

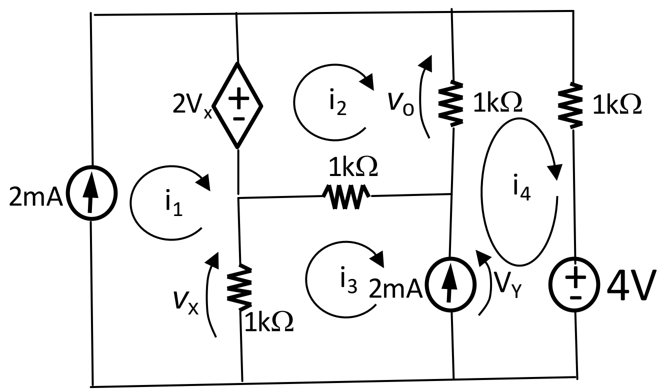
$$144i_1 - 60i_2 = 12V$$

$$144i_1 = 12V + 60(0.1A) = 18V$$

Therefore, i_1 = 0.125A. Thus the net current flow through R_2 is: 0.125A -0.1A = 0.025A and the output voltage is: 0.025A * 60Ω = 1.5V.

Thus the output is much lower than originally designed showing that a voltage divider is only valid if the load takes a small percentage of the current in R_1 and R_2!

Example: Find V_o



First we look at the circuit to see if there are any easy relationships before we start writing equations. We note that:

1) $i_1 = 2mA$
2) $i_4 - i_3 = 2mA \quad \rightarrow \quad i_4 = 2mA + i_3$
3) $V_x = 1k(i_1 - i_3) = 2V - 1ki_3$

We will now use KVL for each loop where we don't know the current:

Loop 1: Current is known $i_1 = 2mA$ so an equation will likely not add any new information.

Loop 2:

$$2V_x - 1k(i_2 - i_4) - 1k(i_2 - i_3) = 0$$

$$4V - 2ki_3 - 1k(i_2 - i_3 - 2mA) - 1k(i_2 - i_3) = 0$$

$$6V - 2ki_3 - 2k(i_2 - i_3) = 0$$

$$6V - 2ki_2 = 0$$

$$i_2 = 3mA$$

Loop 3:

$$V_x - 1k(i_3 - i_2) - V_Y = 0$$

$$2V - 1ki_3 - 1k(i_3 - 3mA) - V_Y = 0$$

$$V_Y = 5V - 2ki_3$$

Loop 4:

$$V_Y + 1k(i_2 - i_4) - 1ki_4 - 4V = 0$$

$$5V - 2ki_3 + 1k(i_2 - i_4) - 1ki_4 - 4V = 0$$

$$1V - 2ki_3 + 1ki_2 - 2ki_4 = 0$$

$$4V - 2ki_3 - 2ki_4 = 0$$

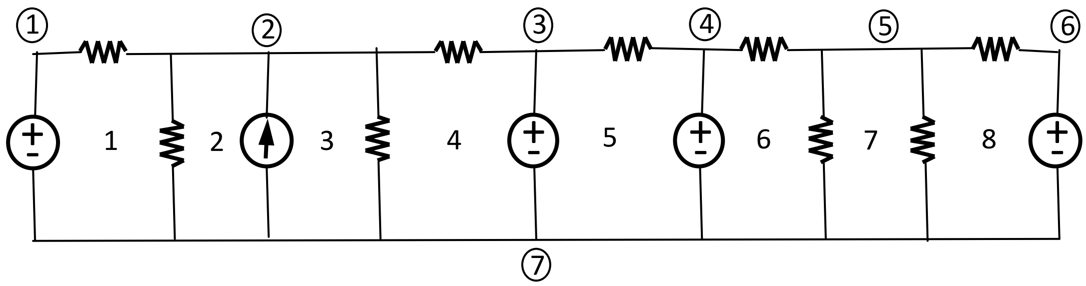
$$4V - 2ki_3 - 2k(2mA + i_3) = 0$$

$$i_3 = 0$$

Therefore $V_Y = 5V$, $i_4 = 2mA$ and finally $V_o = 1k(i_2 - i_4) = 1V$.

Which Method Should I Choose?

Both nodal and loop analysis can be used to completely define circuit variables. Which is best to use? It depends on the circuit.



8 loops -> we might have 1 current fixed by source.

7 nodes -> 4 voltages fixed by sources. Only 2 node equations (node 2, node 5)

Equivalence - we have seen we can represent a circuit by an equivalent -> same behavior e.g. series or parallel resistors wye and delta connections

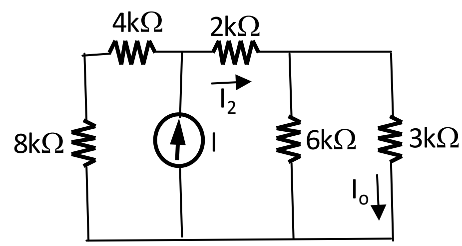
Linearity - our circuits are described by linear algebraic equations.

In the following topics we will learn to break down circuits into analysis that can be combined -> superposition and we will learn to replace circuits with simple equivalents to aid further analysis -> Thevenin and Norton equivalents.

Superposition - any linear circuit containing multiple independent sources, the current or voltage at any point can be calculated as the sum of contributions from each source alone.

Linearity

If a circuit is linear, then any analysis can be linearly scaled -> if input doubles, output also doubles.



If I_o = 1mA, what is I?

If I = 6mA, what is I_o?

By current divider $I_o = 2/3 I_2 = 6k/(3k+6k)\, I_2$

and by current divider once more:

$$I_2 = \frac{4k + 8k}{4k + 8k + 2k + 3k//6k} I$$

$$= \frac{3}{4} I$$

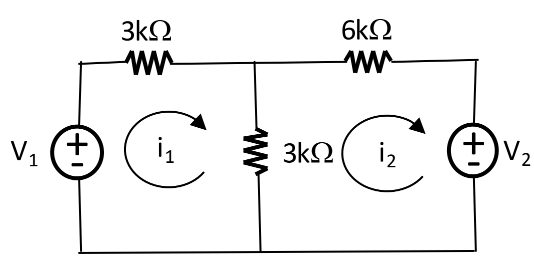
$I_o = 1mA = 2/3 \cdot 3/4 \cdot I \qquad I = 2mA$

Then if I =6mA, I_o = 3 X 1mA = 3mA

We don't have to resolve the entire circuit!

Superposition

Find i_1 and i_2:



Our current approach might be to write mesh (loop) currents:

$3ki_1 + 3k(i_1 - i_2) = v_1$

$6ki_2 + 3k(i_2 - i_1) = -v_2$

Simplifying:

$6ki_1 - 3ki_2 = v_1$

$-3ki_1 + 9k\, i_2 = -v_2$

solving for i_1:

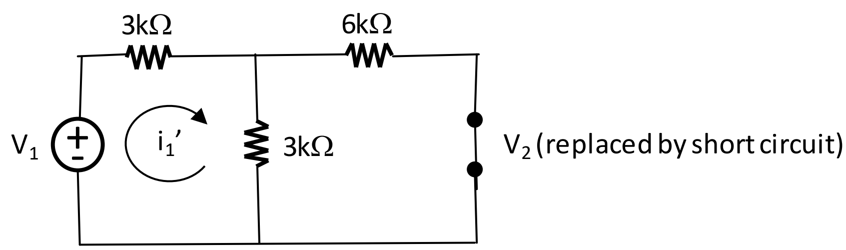
$15ki_1 = 3v_1 - v_2$

$i_1 = V_1/5k - V_2/15k$

Note there are two terms in the answer, one depends on V_1 and one of V_2.

Could we remove one source and analyze?

- ➔ Set $V_2 = 0V$ -> replace with short circuit.

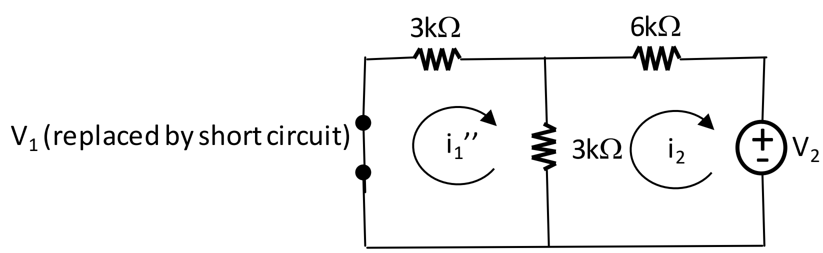


$i_1' = V_1/R_{eq}$

$R_{eq} = 3k + 6k//3k = 5k$

$i_1' = V_1/5k$

also we can remove V_1:



$i_1'' = ½\, i_2$ (simple current divider)

$i_2 = -V_2/R_{eq} = V_2 / (6k + 3k//3k)$

$i_1'' = - ½\, V_2/7.5k = -V_2/15k$

and since the circuit is linear we can add the effects

$i_1 = i_1' + i_1'' = V_1/5k - V_2/15k$

The same result we had before from mesh (loop) analysis.

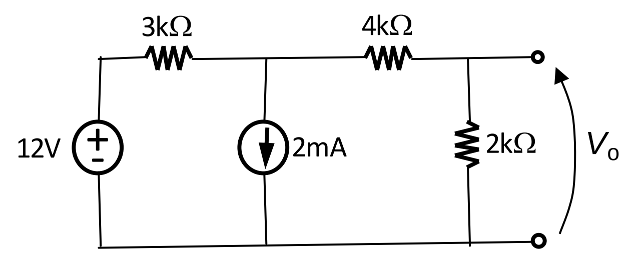
What would we do with a current source?

<u>Current = 0 is an open circuit!</u>

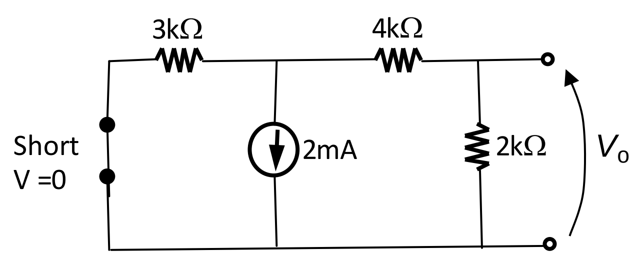
To solve a problem with superposition:

1) Choose a source for analysis.
2) Turn off all other sources -> voltage source = short, current source = open.
3) Complete analysis
4) Repeat for each source -> some might be combined if analysis is straightforward.
5) Sum results to get total.

Example Find V_o:

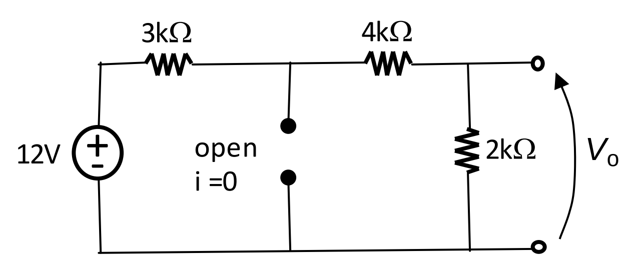


Let's remove the voltage source first and consider only the effect of the current source:



We have a current divider between the 3k resistor and the 4k in series with the 2k resistor. So the fraction of the 2mA current that flows through the 2k resistor is -2mA (3k/(3k+4k+2k) = -0.67mA. Therefore by ohm's Law V_o = -0.67mA X 2k = -1.3V.

Now let's consider the voltage source:



Here we now have a simple voltage divider between 3 resistors so: V_o = 12(2k/(3k+4k+2k) = 2.7V.

Thus the overall voltage is: V_o = 2.7V – 1.3V = 1.4V