Kirchoff's Laws

We have considered some simple 2 resistor circuits and are able to analyze them (determine all electrical variables) using Ohms law and conservation of current, energy and power.

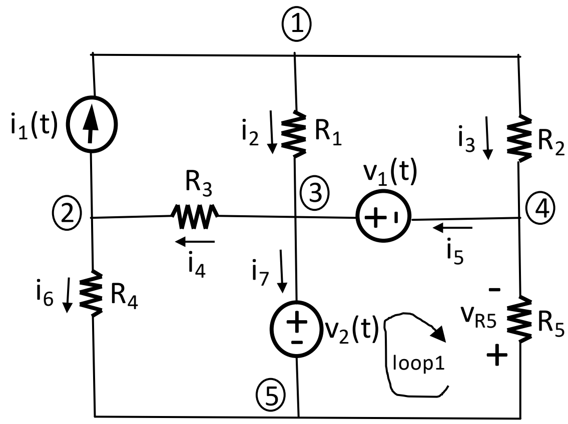
We would like to present some more formal methods to systematically analyze more complex circuits.

First, we need some more terminology.

Node = point of connection of two (or more) circuit elements. Sometimes a node isn't a point but is 'spread out' by ideal wires.

Loop = any closed path through the circuit in which no node is encountered more than once.

Branch = portion of a circuit containing a single element and the nodes at each end of the element.



5 nodes
9 loops
8 branches

Using these definitions we can more formally define conservation of charge in:

Kirchoff's Current Law (KCL)

The algebraic sum of the currents entering any node is zero:

$$\sum_{j=1}^{N} i_j(t) = 0$$

For a node with N branches connected. A current leaving the node would be given the opposite sign to a current entering the node.

Consider node 3 in the previous circuit:

$$i_2(t) + i_5(t) - i_4(t) - i_7(t) = 0$$

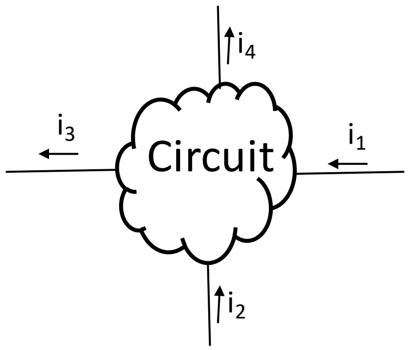
But note we can multiply by -1 with no change to the equation:

$$-i_2(t) - i_5(t) + i_4(t) + i_7(t) = 0$$

In both expressions current in = current out:

$$i_2(t) + i_5(t) = i_4(t) + i_7(t)$$

It is also possible to generalize a bit farther and say the sum of currents entering a closed volume is zero. This isn't used often, but occasionally we want to avoid analyzing part of a circuit, and can look at it as a 'black box'



$$i_1 + i_2 = i_3 + i_4$$

Kirchoff's Voltage Law (KVL)

The algebraic sum of voltages around any loop is zero. This is like walking in a circle, if you do this your net change in altitude is zero.

$$\sum_{j=1}^{N} v_j(t) = 0$$

For a loop with N voltages.

The polarity of the voltage determines the sign.

If your loop enters the positive terminal of a voltage first you can take it as positive, and if your loop enters the negative terminal you can take it as negative.

But you can go around the loop in the opposite direction. All the signs switch, but the equation is the same.

Consider loop 1 of our example circuit. As drawn (clockwise)

$$v_1(t) - v_{R5}(t) - v_2(t) = 0$$

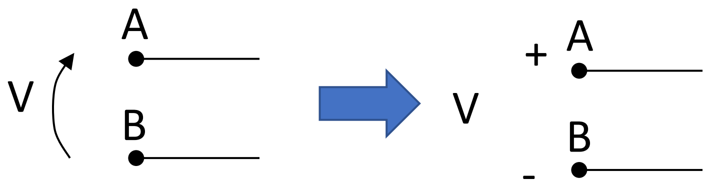
Or going counter clockwise:

$$-v_1(t) + v_{R5}(t) + v_2(t) = 0$$

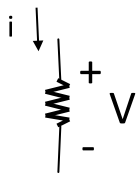
$$v_1(t) - v_{R5}(t) - v_2(t) = 0$$

which leads to the identical result!

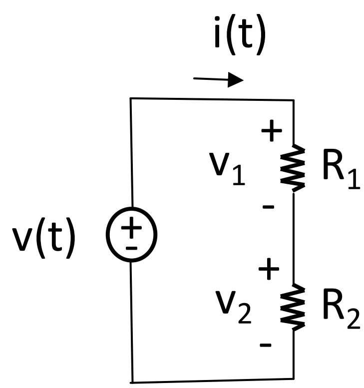
Note that arrows are often used to indicate voltage polarity instead of the + and -:



Also resistors always dissipate energy so always define current and voltage so current enters the positive terminal:



Single Loop Circuits



You can look at any node (how many?) and see the same current flows through each element.

This is a series connection

KVL gives us: $v(t) = v_1 + v_2$

and Ohm's law: $v(t) = i(t)R_1 + i(t)R_2 = i(t)R_{eq}$

where: $R_{eq} = R_1 + R_2$

so series resistors sum:

$R_{eq} = R_1 + R_2 + \cdots + R_N$ for N series resistors.

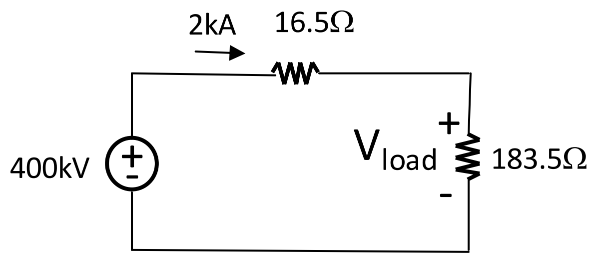
In our 2 resistor circuit the current is given by: $i(t) = \frac{v(t)}{R_1+R_2}$

so the voltage across R_1 is: $v_1 = i(t)R_1 = R_1 \frac{v(t)}{R_1+R_2}$

so the ratio of v_1 to v(t) is: $\frac{v_1}{v(t)} = \frac{R_1}{R_1+R_2}$ which is another way of deriving a voltage divider!

Example

Say we have a 400km long high voltage DC transmission line with a resistance of 0.04125Ω/km. The generating station supplies 400kV at 2kA. The load is resistive with a value of 183.2Ω.



400km X 0.04125Ω/km = 16.5Ω.

What is the power input?

400kV X 2kA = 800MW

What is the power delivered to the load?

$v_{load} = 400kV \frac{183.5}{16.5+183.5}$=367kV by voltage divider.

So the power to the load is 367kV X 2kA = 734MW.

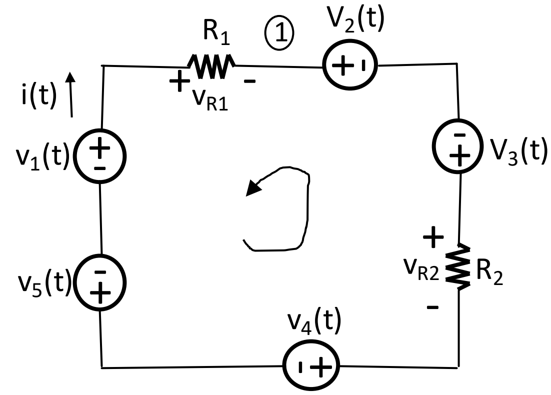
The loss in the line, P_{Loss} = 800mW-734MW = 66MW.

This could also be found by knowing that $P_{Loss} = I^2 R_{line}$.

Since we probably can't reduce R_{Line}, we would like to transmit high voltage, low current to minimize losses.

Multiple Source/Resistor Networks

Provided there is only a single loop, the same current flows in every element (series connection) and the analysis is still straight forward.



We have already noted the series resistance adds. The series voltages also add.

Lets use KVL around the loop, going counter clockwise:

$$v_{R1} - v_1 + v_5 + v_4 + v_{R2} - v_3 + v_2 = 0$$

Grouping voltage sources and resistors:

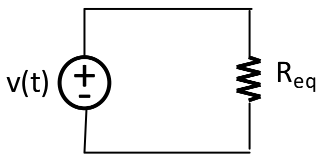
$$[v_{R1} + v_{R2}] + [-v_1 + v_2 - v_3 + v_4 + v_5] = 0$$

Or in other words the net voltage generated by the sources must equal the voltage drop across the resistors:

$$[v_1 - v_2 + v_3 - v_4 - v_5] = [v_{R1} + v_{R2}]$$

$$v_{eq}(t) = v_{Req}$$

Thus the circuit could be simplified:



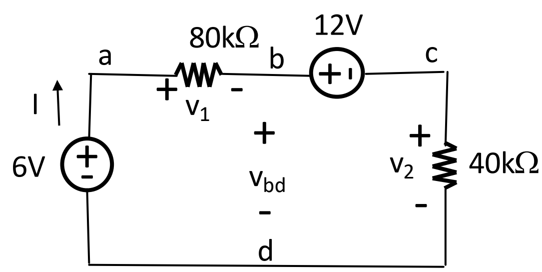
But of course, you would only combine if you didn't need to details on individual elements.

Strategy:

- Define a current (same everywhere)
- Use ohm's law to define resistor voltages
- Apply KVL
- Find the current i(t) -> if it is negative the direction is opposite the definition.

Example

Find the current I and the voltage v_{bd} in the following circuit. Note the notation v_{bd} is read as the voltage from node d to node b.



Apply KVL counter clockwise: v_2+12V+v_1 – 6V = 0

Use Ohm's Law to write voltages in terms of current: I ·40k +12 + I ·80k -6 = 0

Solve for I: I ·120k = -12+6 -> I = -6/120k = -50μA (0.05mA) note the negative sign means the current is flowing in the opposite direction to shown.

To find v_{bd} we can do KVL around a loop that includes only v_{bd}, v_1 and the 6V supply: v_1 -6+v_{bd} = 0 and since we already know the current flowing through the 80k resistor:

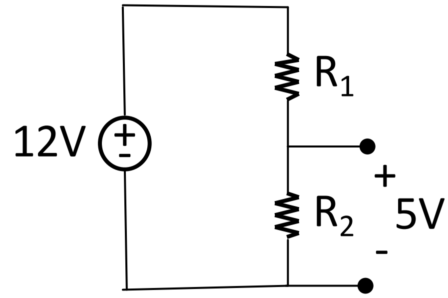
v_{bd} = 6-(-50μA) ·80k = 10V

Example

Design a circuit to produce a 5V output from a 12V input. The power consumption should not exceed 240mW.

This sounds like a voltage divider:

$$5V = 12V\frac{R_2}{R_1 + R_2}$$



But there are two unknown resistors.

We also know:

$$P = \frac{V^2}{R} = \frac{(12V)^2}{R_1 + R_2} \leq 0.24W$$

$$R_1 + R_2 \geq \frac{144}{0.24} \geq 600$$

Therefore:

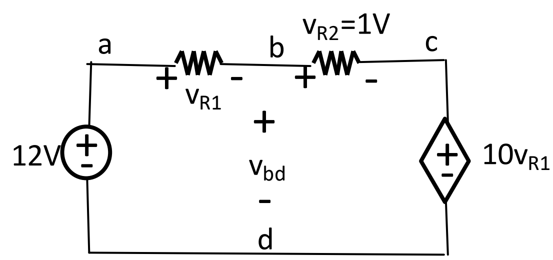
$$\frac{R_2}{600} \geq \frac{5}{12}$$

So $R_2 \geq 250\Omega$ and $R_1 \geq 350\Omega$.

It may not be possible to find these exact values, but close is often good enough in circuit design.

<u>Example</u>

Find v_{bd} in the following circuit:



Once more we apply KVL to the loop: $12V - V_{R1} - 1V - 10V_{R1} = 0$

Noting there is only one unknown: $11V_{R1} = 11V$ -> $V_{R1} = 1V$

It should be easy to see from inspection that therefore $v_{bd} = 10V + 1V = 11V$ (the drop across the resistor plus the drop across the voltage-controlled voltage source).

So to summarize

Ohm's Law

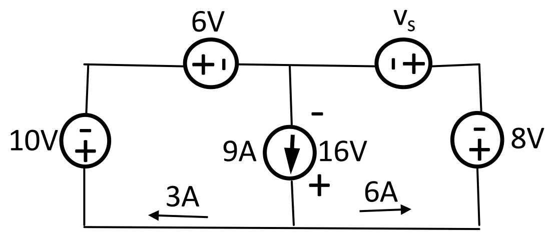
Kirchoff's Laws KCL

KVL

Single Loop -> KVL

➔ Voltage division

Note power must also be conserved (though not used as often as KCL, KVL)



Is v_s supplying or absorbing power? How much?

KVL -> 8V +v_s -16V = 0 -> v_s = 8V Ps = 8V X 6A = +48W absorbed.

Or look at the power for the entire circuit.

3A X 10V = 30W

3A X 6V = 18W

-9A X 16V = -144W

6A X 8V = 48W

6A X v_s = Ps

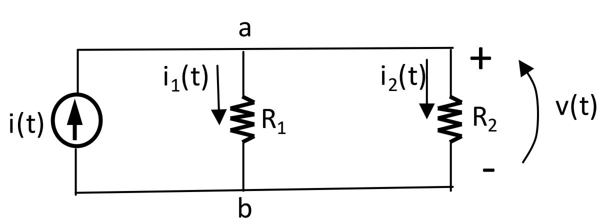
$$\sum P = 0 = 30 + 18 + 48 + P_s - 144$$

Ps = 144-96=48W

V_s =48W/6A = 8V

Single Node Pair Circuits

Lets now add a loop to our circuit, but for now limited to 2 nodes.



All elements have the same voltage across them

A parallel (or perhaps shunt) combination

Consider charge conservation, now KCL:

$$i(t) = i_1(t) + i_2(t)$$

$$= \frac{v(t)}{R_1} + \frac{v(t)}{R_2}$$

$$= \left(\frac{1}{R_1} + \frac{1}{R_2}\right) v(t)$$

$$= \frac{v(t)}{R_{eq}}$$

Where $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$ or $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

In general:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

For N parallel resistors. Note as more resistors are added in parallel the overall equivalent resistance is reduced.

How does the current divide between the two resistors in our circuit?

If $v(t) = R_{eq} i(t) = \frac{R_1 R_2}{R_1 + R_2} i(t)$

and $i_1(t) = \frac{v(t)}{R_1}$

then $i_1(t) = \frac{R_2}{R_1 + R_2} i(t)$ and $i_2(t) = \frac{R_1}{R_1 + R_2} i(t)$

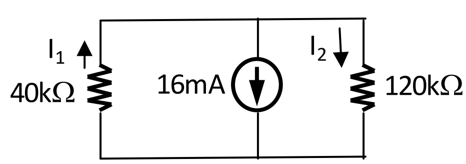
written in an alternate form: $i_1(t) = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} i(t) = \frac{i(t)}{R_1} R_{eq} = i(t) \frac{R_{eq}}{R_1}$

similarly for R_2: $i_2(t) = i(t) \frac{R_{eq}}{R_2}$

This is just another way of deriving a current divider. We note as well that current from the source is divided in proportion to the inverse of the resistance.

<u>Example</u>

Find I_1 and I_2 and the power absorbed by the 40kΩ resistor:



KVL: $I_1 \cdot 40k\Omega + I_2 \cdot 120k\Omega = 0$ -> $I_1 + 3I_2 = 0$ (1)

KCL: $I_1 - 16mA - I_2 = 0$ -> $I_1 - I_2 = 16$ (2)

Now we have two equations and two unknowns. Lets start by subtracting (2) from (1) to remove I_1:

$4I_2 = -16$ -> $I_2 = -4mA$ (which indicates that I_2 will actually flow in the opposite direction to that indicated on the drawing). Knowing I_2 we can now find $I_1 = 16 + (-4) = 12mA$.

Alternatively, we could have noted that this circuit is once more a simple current divider:

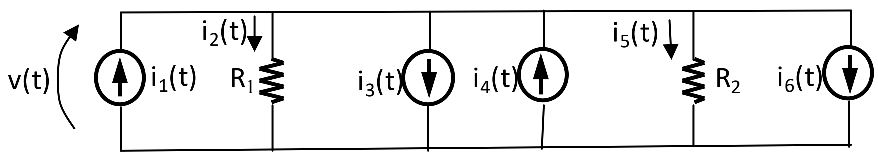
$$I_1 = \frac{120k}{120k + 40k} 16mA = 12mA$$

$$I_2 = \frac{-40k}{120k + 40k} 16mA = -4mA$$

Finally, the power in the 40k resistor can be calculated as: $P = I^2R = (12mA)^2 X 40k = 5.76W$.

Multiple Sources and Resistors

We can apply the same techniques to more complex circuits, but we need to be careful with signs.

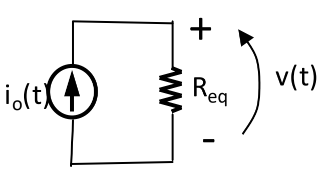


Apply KCL to the upper node:

$$i_1(t) - i_2(t) - i_3(t) + i_4(t) - i_5(t) - i_6(t) = 0$$

$$i_1(t) - i_3(t) + i_4(t) - i_6(t) = i_2(t) + i_5(t)$$

current sources sum = resistor current sum



$$i_o(t) = \frac{v(t)}{R_{eq}} = v(t)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

and if we had more resistors:

$$i_o(t) = v(t)\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}\right)$$

Therefore:

$$\frac{1}{R_{eq}} = \sum_{i=1}^{N} \frac{1}{R_i} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}\right)$$

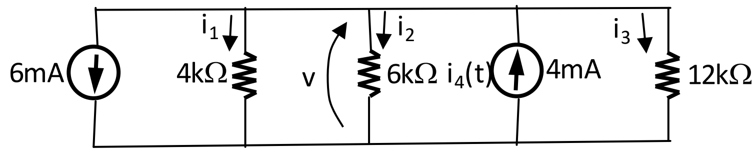
For a parallel combination of resistors.

Strategy

- Define a voltage between two nodes (including polarity)
- Use ohm’s law to define a current through each resistor in terms of voltage
- Apply KCL at one of the nodes
- Solve the resulting equation for v(t). sign of result determines if the polarity was correct.

Example

Find the power absorbed by the 6kΩ resistor.



Net current from the current sources is 2mA (flowing down).

Net resistance is: 4k//6k//12k = 2k

Voltage across the resistors is v = 2mA * 2k = 4V

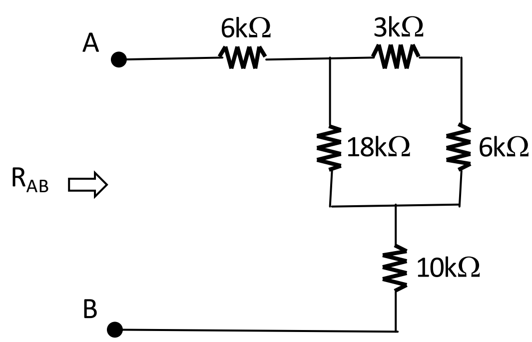
Therefore the power through the 6k resistor is: $P = V^2/R = 16/6k = 2.67mW$

Series and Parallel Combinations

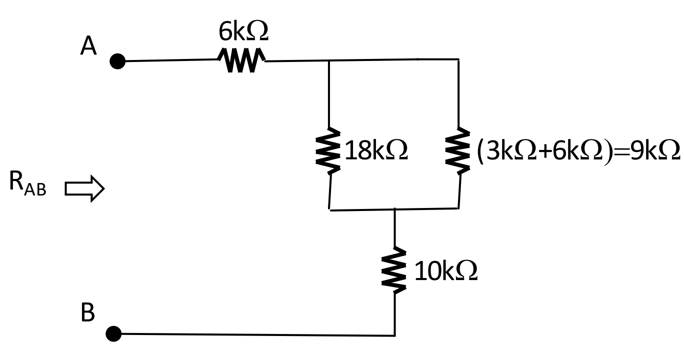
We now know the equivalent resistance of some number of series resistors or parallel resistors.

Most networks are a combination of resistors in series and parallel. The trick is to learn to see this, visualize it and then simplify the circuit.

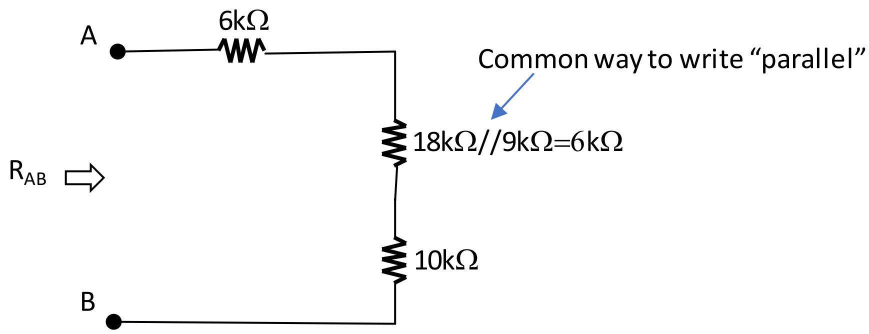
Find the equivalent resistance between A and B:



What can be combined? -> 3k and 6k are pure series:



Now 18k and 9k are in parallel:

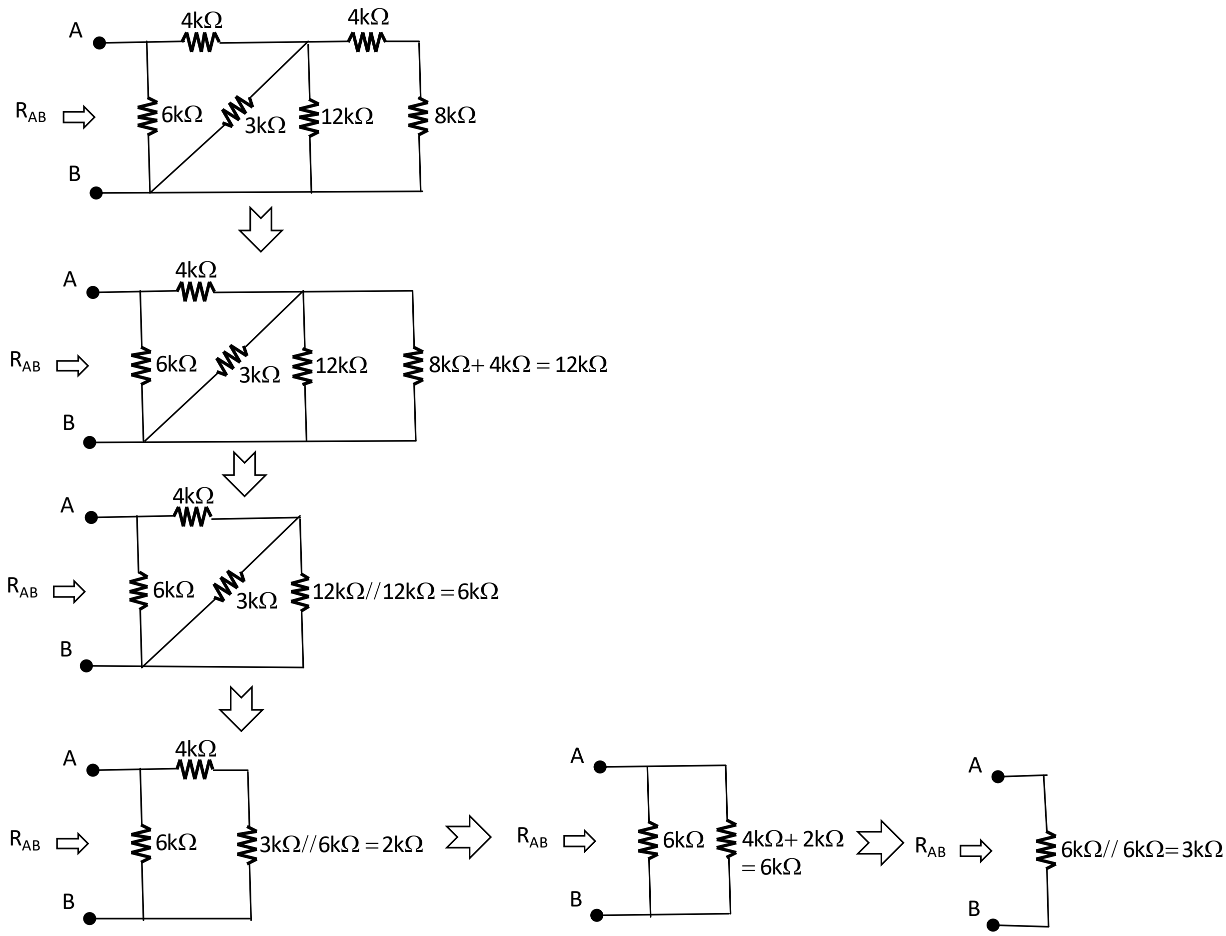


And now we have a straight series combination:

R_{AB} = 6k+6k+10k = 22k

You may have to repeat steps a few times to get to a single equivalent resistor.

Series -> parallel -> series -> parallel …..



Resistor Specification

Key parameters are value, tolerances, and power rating

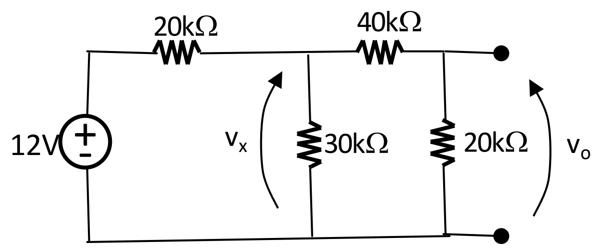
One of the most common failures in electronics is exceeding the power rating for a resistor resulting in the failure of the resistor and perhaps connected parts

We usually use 1/4W resistors in the lab, but larger ratings are necessary in many applications.

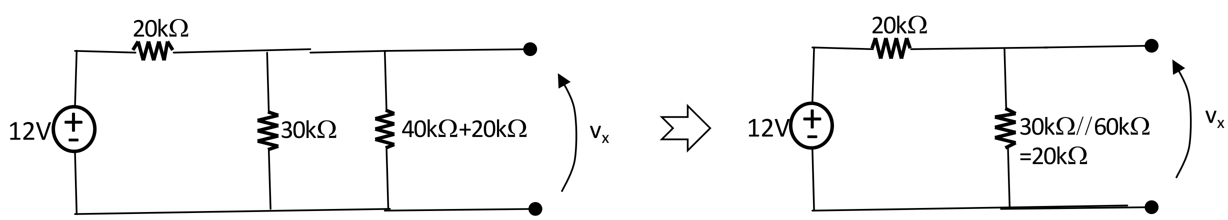
Standard values of resistance are available for each decade of value (10X) depending on tolerance. Standard values are chosen such that all values in the range are possible given the tolerance. e.g. a 4.7kΩ resistor with a 10% tolerance could have an actual value from 4.23kΩ to 5.17kΩ which overlaps with the range for a 3.9kΩ and 5.6kΩ resistor. So smaller tolerance resistor series need to include more values.

Example:

Find v_o:



There is typically more than one way to a solution…



Then v_x can be found from a voltage divider:

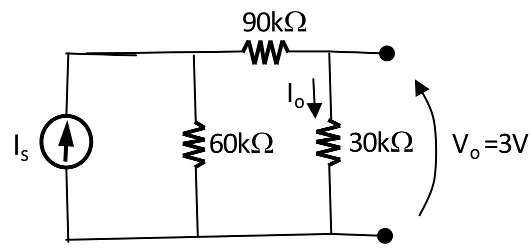
$$v_x = 12V\frac{20k}{20k + 20k} = 6V$$

And v_o can be found from a second voltage divider:

$$v_o = v_x\frac{20k}{40k + 20k} = 2V$$

Example:

Find a source current Is that will provide a designed output of 3V.



To get 3V across the 30kΩ resistor, we require I_o X 30kΩ = 3V or I_o = 0.1mA.

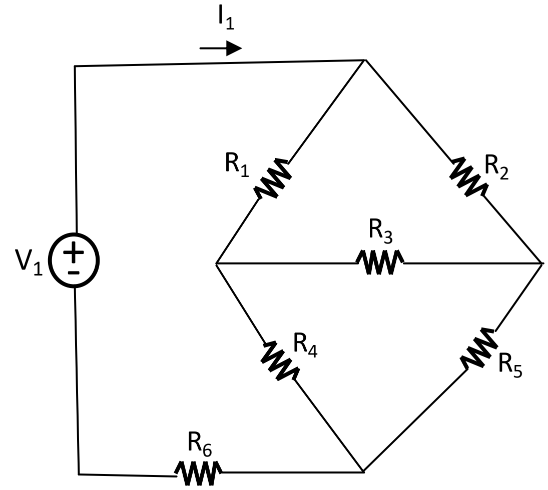
This is divided from the source current Is:

$I_o = I_s \frac{R_{eq}}{120k}$ * the current splits between the branch with 60k and the branch with 30+90=120k.

Where $R_{eq} = \frac{1}{\frac{1}{60k}+\frac{1}{120k}} = 40k$

So $I_s = I_o \frac{120k}{40k} = 0.3mA$

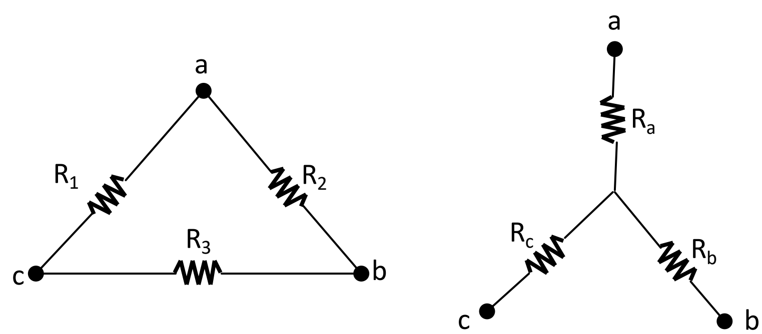
Wye-Delta Transformations



A 'bridge' circuit is often used in sensor electronics but notice we cannot combine resistors in the manner we have learned so far. However, it is possible to transform parts of this to another equivalent form, to ease analysis.

This wye-delta configuration change is most commonly used in power circuits, which have 3 different sinusoidal sources connected in the wye (Y) or delta (Δ) configuration.

We can find a transformation from delta to wye:



By making the resistance between two terminals equal:

$$R_{ab} = R_a + R_b = (R_1 + R_3)//R_2 = \frac{R_2(R_1 + R_3)}{R_1 + R_2 + R_3}$$

$$R_{bc} = R_b + R_c = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3}$$

$$R_{ca} = R_c + R_a = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$

Which gives us 3 equations to solve for R_a, R_b, and R_c in terms of R_1, R_2, and R_3.

And the end result is:

$$R_a = \frac{R_1 R_2}{R_1 + R_2 + R_3}$$

$$R_b = \frac{R_2 R_3}{R_1 + R_2 + R_3}$$

$$R_c = \frac{R_1 R_3}{R_1 + R_2 + R_3}$$

Or:

$$R_1 = \frac{R_aR_b + R_bR_c + R_cR_a}{R_b}$$

$$R_2 = \frac{R_aR_b + R_bR_c + R_cR_a}{R_c}$$

$$R_3 = \frac{R_aR_b + R_bR_c + R_cR_a}{R_a}$$

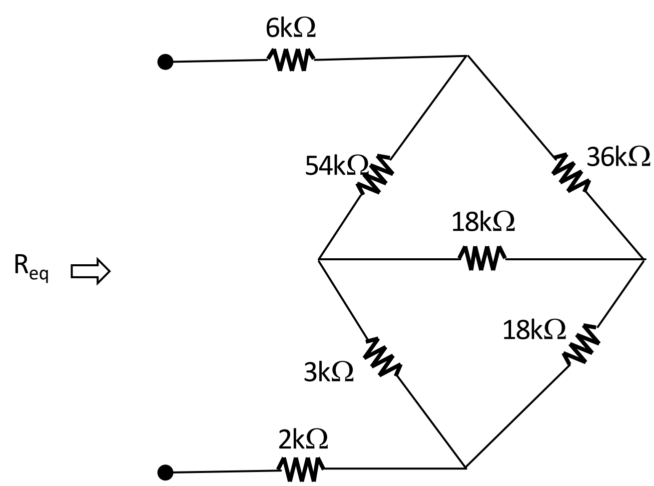
For the 'balanced' case R_a=R_b=R_c, then R_1=R_2=R_3

$$R_Y = \frac{1}{3}R_\Delta \text{ or } R_\Delta = 3R_Y$$

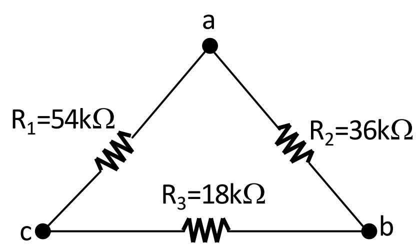
How would you use this? Most often in 3ϕ power with balanced loads.

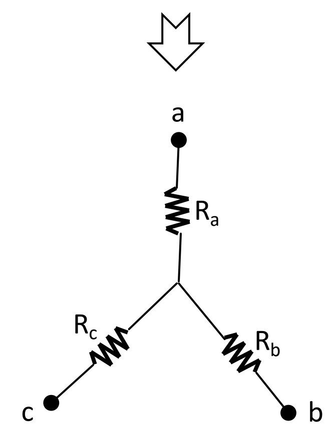
Example

Find the equivalent R_{eq}:



Consider the delta part of the circuit first:





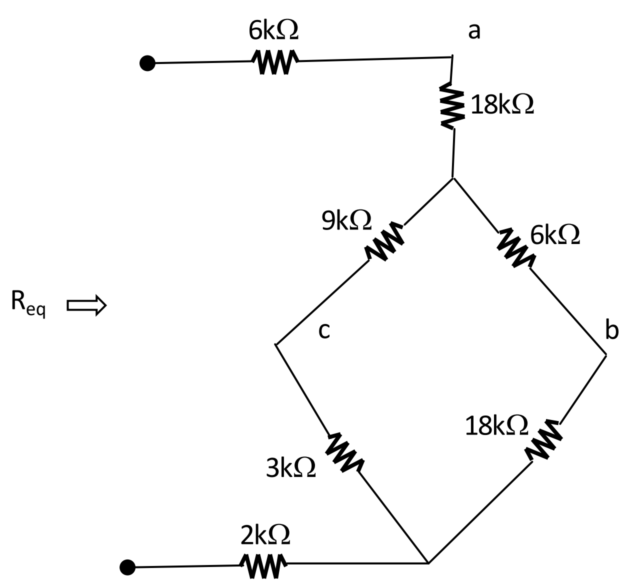
Using the formulas just derived:

R_a = 54X36/108 = 18kΩ

R_b = 36X18/108 = 6kΩ

R_c = 54X18/108 = 9kΩ

Now we can redraw the circuit using the delta equivalent:



Now from inspection: Req = 6k+18k+2k+24k//12k = 34k